Monetary policy and redistribution: What can or cannot be neutralized with Mirrleesian taxes

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Abstract

This paper develops an overlapping-generations model with heterogeneous agents in terms of earning ability and cash-in-advance constraint. It shows that tax policy cannot fully replicate or neutralize the redistributive implications of monetary policy. While who gets the extra money becomes irrelevant, the rate of growth of money supply keeps its bite. A second lesson is that it may be optimal to deviate from the Friedman rule. The results are due to the existence of another source of heterogeneity among individuals besides differences in earning ability that underlies the Mirrleesian approach to optimal taxation. They hold even in the presence of a general nonlinear income tax and preferences that are separable in labor supply and goods. If differences in earning ability were the only source of heterogeneity, the fiscal authority would be able to neutralize the effects of a change in the rate of monetary growth and a version of the Friedman rule becomes optimal.

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1 Introduction

This paper attempts to shed light on two inter-related questions. One is the redistributive properties of monetary policy in a model where the fiscal authority is able to levy nonlinear taxes. Specifically, it examines if all redistributive aspects of monetary policy can be replicated, or neutralized, through fiscal policy (ignoring macroeconomic issues). The question is important not only in its own right but also to the resolution of the debate regarding the impropriety of giving redistributive power, which should reside with the legislature, to unelected central bankers. The second question concerns the much debated issue of the optimality of Friedman rule of setting the nominal interest rate to zero. The two questions are related in that the monetary policy affects redistribution through the monetary growth rate as well as money disbursement rule.

Two recent papers have advanced our knowledge on both of these fronts. Williamson (2008) makes a distinction between "connected" and "unconnected" agents in terms of their access to financial institutions. He shows that this source of heterogeneity causes monetary policy to have significant redistributive implications. Additionally, it often leads to a negation of Friedman rule. However, Williamson does not allow for a tax authority with the power to levy nonlinear taxes. da Costa and Werning (2008), on the other hand, allow for nonlinear income taxes and find that Friedman rule is optimal. In their model, however, the source of heterogeneity between agents is something quite different from Williamson's. Their heterogeneity arises from the variation in the earning abilities of different individuals that forms the cornerstone of the Mirrleesian framework. The current paper draws on both of these papers bringing them together in a unified framework. We show that the ability to levy nonlinear taxes can neutralize monetary policy only if the source of heterogeneity concerns earning abilities, as in da Costa and Werning (2008), but not if it concerns heterogeneity of the type Williamson (2008) has in mind.

To put the importance of the Mirrleesian framework in perspective, recall that the

Friedman rule is a first-best prescription and may or may not hold in second-best settings. This depends on the nature of the second-best (existence of distortionary taxes or intrinsic reasons for market failure), the set of tax instruments available to the government, and the structure of individuals' preferences.[1] Chari *et al.* (1991, 1996), in the context of a model with identical and infinitely-lived individuals, related the optimality of Friedman rule in the presence of distortionary taxes to the uniform commodity tax result of Atkinson and Stiglitz (1972) and Sandmo (1974). This latter result states that if preferences are separable in labor supply and non-leisure goods, with the subutility for goods being homothetic, optimal commodity taxes are proportionately uniform. Deviations from Friedman rule violates this tax principle.[2]

The optimality of Friedman rule has traditionally been studied in environments with identical individuals. Such environments are, by construct, silent on the validity of Friedman rule when monetary policy has redistributive implications.[3] Naturally too, these studies which use the Ramsey tax framework, assume that all tax instruments, including the income tax, are set proportionally.[4] The novelty of da Costa and Werning (2008) is that they break with this tradition.[5] They consider a model in which individuals are heterogeneous with respect to their earning ability, and allow the government to levy nonlinear income taxes. Their result too is interesting as they are able to prove that the Friedman rule is optimal for *any* social welfare function that redistributes from

[1] Non-optimality of Friedman rule in the presence of distortive taxes was first discussed by Phelps (1973). A selective reference to other sources of distortion include: van der Ploeg and Alogoskoufis (1994) for an externality underlying endogenous growth; Ireland (1996) for monopolistic competition; Erceg *et al.* (2000) and Khan *et al.* (2003) for nominal wage and price settings; Schmitt-Grohe and Uribe (2004a,b) for imperfections in the goods market; and Shaw *et al.* (2006) for imperfect competition as well as externality.

[2] This uniformity result is derived within the context of the traditional one-consumer Ramsey problem. As such, the result embodies only efficiency considerations. Redistributive goals do not come into play.

[3] With the exception of intergenerational redistributive issues that arise in overlapping generations models; see, e.g., Weiss (1980), Abel (1987), and Gahvari (1988).

[4] See, e.g., Chari *et al.* (1991, 1996), Correia and Teles (1996, 1999), Guidotti and Vegh (1993), and Mulligan and Sala-i-Martin (1997).

[5] See also Albanesi (2007).

the rich to the poor.

As with Chari *et al.*'s (1991, 1996) earlier result, da Costa and Werning's (2008) finding is also related to the uniform taxation result in public finance, albeit a different one. Whereas Chari *et al.*'s (1991, 1996) draws on Sandmo's tax uniformity (1974) result derived within a Ramsey setting, da Costa and Werning's (2008) has its roots in Atkinson and Stiglitz (1976). That classic paper on the design of tax structures was particularly concerned with the usefulness of commodity taxes in the presence of a general income taxes in many-consumer economies.[6] It proved that with a *general* income tax, if preferences are weakly separable in labor supply and goods, then commodity taxes are not needed as instruments of optimal tax policy. With non-separability, one wants to tax the goods that are "substitutes" with labor supply and subsidize those that are "complements" with labor supply. In da Costa and Werning (2008) the uniformity result, which implies a zero nominal interest rate, holds with preference separability. However, da Costa and Werning assume that real cash balances and labor supply are complements so that cash balances should be subsidized. This implies that the optimal nominal interest rate is negative. But given the non-negativity of nominal interest rate, the zero interest rate emerges as the "optimal" policy.

da Costa and Werning's complementarity assumption tells us that *if* a high-ability consumer and a low-ability consumer have the same gross-of-tax income and the same net-of-tax income, the high-ability consumer who will work less (because his wage rate is higher) will carry a smaller amount of real cash balances than the low-ability consumer. However, the assumption does *not* tell us if, in equilibrium, a high-ability person will in fact carry a smaller amount of real cash balances, as a percentage of his total expenditures, than a low-ability consumer. If anything, with a shopping-time rationalization

[6] The ineffectiveness of commodity taxes and their proportionately uniform tax treatment boil down to the same thing. In the absence of exogenous incomes, the government has an extra degree of freedom in setting its income and commodity tax instruments. This is because all demand and supply functions are homogeneous of degree zero in consumer prices and lump-sum income. In consequence, the government can, without any loss of generality, set one of the commodity taxes at zero (i.e. set one of the commodity prices at one). Under this normalization, uniform rates imply absence of commodity taxes.

for money holdings, one may very well expect the reverse of this, as the opportunity cost of time is higher for high-ability individuals. Yet, as Albanesi (2007) argues, the empirical observations show that lower income consumers do carry a higher percentage of their expenditures in cash.[7] This raises two questions. Why is this the case and what are its implications for optimal monetary policy and the Friedman rule?

This paper is not concerned with question of why. Yet it is not too difficult to realize that the answer cannot lie primarily in the heterogeneity of agents with respect to their earning ability (which is the cornerstone of the optimal tax literature). As argued by Williamson (2008), different agents may have to carry different levels of cash balances because of their different levels of access to other financial instruments and/or their sophistication. These, in turn, may be negatively correlated with one's earning ability. These considerations do not arise naturally from da Costa and Werning's complementarity assumption and must be explicitly accounted for.

This paper, following da Costa and Werning (2008), uses a Mirrleesian approach and allows for individuals to have different earning abilities and face a nonlinear income tax schedule. To capture the second source of heterogeneity, it uses a Clower cash-in-advance constraint to rationalize money holdings while allowing for the cash-in-advance reserve requirement to differ across earning abilities. This difference may have arisen from Williamson's (2008) distinction between "connected" and "unconnected" agents in terms of their access to financial institutions. Our setup differs from da Costa and Werning's (2008) in one other important aspect. Ours uses an overlapping-generations framework rather than an infinitely-lived cohort of agents.

One lesson of this paper is that fiscal policy cannot fully replicate or neutralize the redistributive implications of monetary policy. While who gets the extra money becomes irrelevant, the rate of growth of money supply keeps its bite. A second related lesson is that the Friedman rule may be suboptimal even in the face of an optimal nonlinear

[7] She also argues that the complementarity "assumption would lead to a cross-sectional distribution of money holdings that is inconsistent with empirical evidence" (p. 38).

income tax. The reason for both of these results is the existence of other sources of heterogeneity among individuals besides differences in earning ability that underlies the Mirrleesian approach to optimal taxation. If differences in earning ability were the only source of heterogeneity in the model, the fiscal authority would be able to neutralize the effects of a change in the rate of monetary growth and a version of the Friedman rule becomes optimal.[8]

2 The model

Consider a two-period overlapping-generations model where individuals work in the first period and consume in both. There is no bequest motive. Preferences are represented by the strictly quasi-concave utility function $U = u(c_t, d_{t+1}, L_t)$ where c denotes consumption in the first period, d consumption in the second period, and L denotes the labor supply; subscript t denotes calendar time. The utility function is strictly increasing in c_t and d_{t+1}, and strictly decreasing in L_t. Each generation consists of two types of individuals who differ in two correlated characteristics: skill levels (labor productivity) and the "degree of connectedness". High-skilled workers are paid w_t^h and low-skilled workers w_t^ℓ; with $w_t^h > w_t^\ell$. The degree of connectedness is modeled by the relative size of the cash one has to carry for financing his transactions. The proportion of agents of type j, $j = h, \ell$, remains constant over time. Denote this proportion in a given generation by π^j. Population grows at a constant rate, g; with N_t being the total number of agents born in period t. Thus, denoting the total number of agents of type j born in period t by n_t^j, one has $\pi^j = n_t^j / N_t$.

Production takes place through a linear technology with different types of labor as inputs. Transfer of resources to the future occurs only through a storage technology with a fixed (real) rate of return, r.[9] We thus work with an overlapping-generations model à

[8] More precisely, the Friedman rule is not a unique optimum; a continuum of values for the monetary growth rate and the tax on the second-period consumption maximizes social welfare.

[9] An alternative assumption is that agents borrow and lend on international capital markets at an

la Samuelson (1958) and assume away the issues related to capital accumulation.

2.1 Money and monetary policy

Money holdings, rationalized through a Clower cash-in-advance constraint, constitute another source of financing for future consumption (in addition to real savings). At the beginning of period t, before consumption takes place, the young purchase all the existing stock of money, M_t, from the old. Denote a young j-type agent's purchases by m_t^j. We have

$$M_t = n_t^h m_t^h + n_t^\ell m_t^\ell. \tag{1}$$

The rate of return on money holdings (the nominal interest rate), i_{t+1}, is related to the inflation rate, φ_{t+1}, according to Fisher equation

$$1 + i_{t+1} \equiv (1 + r_{t+1}) \left(1 + \varphi_{t+1}\right). \tag{2}$$

Denote the price level at time t by p_t; the inflation rate is defined as

$$1 + \varphi_{t+1} \equiv p_{t+1}/p_t. \tag{3}$$

The monetary authority injects money into (or retires money from) the economy at the constant rate of θ. Money is given to (or taken from) the old—who hold all the stock of money—via lump-sum monetary transfers (or taxes). Thus a young j-type agent who purchases m_t^j at the beginning of time t receives a_{t+1}^j at the beginning of period $t+1$. Clearly, a_{t+1}^h and a_{t+1}^ℓ must satisfy the "money injection relationship",

$$n_t^h a_{t+1}^h + n_t^\ell a_{t+1}^\ell = \theta M_t. \tag{4}$$

Beyond this, we do not specify how much of the extra money injection goes to which type. Indeed, an important message of our paper is to prove that this division is immaterial.

exogenously fixed interest rate.

With money stock changing at the rate of θ in every period, $M_{t+1} = (1+\theta) M_t$. Substitute for M_t and M_{t+1}, from equation (1), into this relationship:

$$n^h_{t+1} m^h_{t+1} + n^\ell_{t+1} m^\ell_{t+1} = (1+\theta) \left(n^h_t m^h_t + n^\ell_t m^\ell_t \right).$$

Given that the population of each type grows at the constant rate of g, one can rewrite this as[10]

$$n^h_t \left(m^h_{t+1} - \frac{1+\theta}{1+g} m^h_t \right) + n^\ell_t \left(m^\ell_{t+1} - \frac{1+\theta}{1+g} m^\ell_t \right) = 0.$$

Assume that the money-holdings of each type changes in the same direction. It then follows that

$$m^j_{t+1} = \frac{1+\theta}{1+g} m^j_t. \tag{5}$$

Following Hahn and Solow (1995), specify the cash-in-advance constraint through the assumption that all agents must finance a fraction of their second-period consumption expenditures by the cash balances saved in the first period.[11] However, given our heterogeneous-agents framework, this fraction is not the same for individuals of different types. Specifically, let γ denote the fraction of one's second-period consumption expenditures that has to be financed by cash balances. Given Williamson's (2008) notion of connectedness, one would expect that γ depends negatively on skills: The more skilled individuals, being more sophisticated and more connected, require a smaller amount of cash to finance their transactions. Additionally, to account for the empirical observation that lower income individuals carry a higher amount of cash relative to their expenditures as stated by Albanesi (2007), we assume that γ also depends negatively on one's

[10] Observe that $(1+g)\, m^j_{t+1}$ is not necessarily equal to $m^j_t + a^j_{t+1}$. This will be the case if $a^j_{t+1} = \theta m^j_t$.

[11] This specification has been used extensively in overlapping-generations models, particularly by Philippe Michel and his associates; see, e.g., Crettez *et al.* (1999, 2002) and Michel and Wigniolle (2003, 2005). This specification may appear restrictive in that it does not apply to first-period consumption expenditures. However, this is *not* the case for the points addressed in this paper. Assuming that first-period expenditures are also subject to this constraint does not change our results. Given that individuals have no assets in the first-period, they will have to borrow money from the old, at the market interest rate, and as such imposes no additional constraint on the individuals' optimization problem. See Gahvari (2012) for more details on what might change if one adopts this more generalized specification for the cash-in-advance constraint.

income, either gross income I, or aggregate disposable income y.[12] Using γ^j to denote either $\gamma\left(w^j, I^j\right)$ or $\gamma\left(w^j, y^j\right)$ (depending on whether γ is assumed to depend on gross- or aggregate disposable income), one can write the j-type's cash-in-advance constraint by

$$m_t^j + a_{t+1}^j \geqq \gamma^j p_{t+1} d_{t+1}^j. \tag{6}$$

Assume constraint (6) binds. Dividing it by p_{t+1}, rearranging the terms, and using equations (2) and (3), yields

$$\begin{aligned} \frac{m_t^j}{p_{t+1}} &= \gamma^j d_{t+1}^j - \frac{a_{t+1}^j}{p_{t+1}}, \\ &= \gamma^j d_{t+1}^j - \frac{a_{t+1}^j}{p_t} \frac{1+r_{t+1}}{1+i_{t+1}}. \end{aligned} \tag{7}$$

2.2 Fiscal policy

The tax authority is able to levy income and commodity taxes. Assume, in the tradition of the optimal income tax literature à la Mirrlees (1971), that an individual's type and labor supply are not publicly observable but that his labor income, $I_t = w_t L_t$, is. This rules out first-best type-specific lump-sum taxes but allows labor income to be taxed via a general (nonlinear) tax schedule $T(I_t)$. Assume further that the information the tax authority has on transactions, including money holdings, is of anonymous nature; it does not know the identity of purchasers. This assumption, which is made for realism, implies that goods can be taxed only linearly (possibly at different rates). Appendix F explores the implication of allowing the government to have information on individual purchases.

As usual, homogeneity of degree zero of demands in consumer prices, and supplies in producer prices, allows one to normalize both sets of prices. This enables us to normalize

[12] This is a more general specification than allowing for γ to depend on income only indirectly through one's skill level. It seems reasonable, and in line with Williamson's argument, that one's level of income accords him a measure of connectedness regardless of his innate skill level.

one of the commodity tax rates to zero. We set the tax rate on c_t to be zero and denote the tax rate on d_t by τ. All producer prices are normalized to one.

2.3 Constrained Pareto-efficient allocations

To characterize the (constrained) Pareto-efficient allocations, one has to account for the economy's resource balance, the standard incentive compatibility constraints due to our informational structure, and the implementability constraints caused by linearity of commodity taxes—itself due to informational constraint, as well as the monetary expansion mechanism. To this end, we derive an optimal revelation mechanism. For our purpose, a mechanism consists of a set of type-specific before-tax labor incomes, I_t^j's, after-tax incomes, z_t^j's, a commodity tax rate, τ, a money supply growth rate, θ, and a monetary distributive rule, a_t^j. This procedure determines τ, θ, and a_{t+1}^j from the outset. A complete solution to the optimal tax problem per-se, i.e. determination of I_t^j by the individuals via utility maximization, then requires only the design of a general income tax function $T(I_t)$ such that $z_t^j = I_t^j - T\left(I_t^j\right)$.

To proceed further, it is necessary to consider the optimization problem of an individual for a given mechanism $(\tau, \theta, a_{t+1}, z_t, I_t)$. This is necessitated by the fact that the mechanism determines personal consumption levels only indirectly, namely through prices. The mechanism assigns the quintuple $\left(\tau, \theta, a_{t+1}^j, z_t^j, I_t^j\right)$ to a young individual who reports his type as j. The individual will then allocate z_t^j between first- and second-period consumption, and money holdings.

Formally, given any vector $(\tau, \theta, a_{t+1}, z_t, I_t)$, an individual of type j chooses c_t and d_{t+1} to maximize

$$u = u\left(c_t, d_{t+1}, \frac{I_t}{w_t^j}\right), \quad j = h, \ell, \tag{8}$$

subject to the per-period budget constraints

$$p_t (c_t + s_t) + m_t = p_t z_t, \tag{9}$$

$$p_{t+1} (1 + \tau) d_{t+1} = p_t s_t (1 + i_{t+1}) + m_t + a_{t+1}, \tag{10}$$

where s_t is the level of real savings chosen by the agent. Observe that θ does not explicitly appear in the problem above; it does so implicitly thorough its effect on i_{t+1}. Equations (9)–(10) can be unified into an intertemporal budget constraint for the young. Substitute $z_t - c_t - m_t/p_t$ for s_t from (9) into (10) to derive,

$$\begin{aligned} p_{t+1} (1 + \tau) d_{t+1} &= p_t \left(z_t - c_t - \frac{m_t}{p_t} \right) (1 + i_{t+1}) + m_t + a_{t+1} \\ &= p_{t+1} \left[z_t - c_t - \frac{m_t}{p_{t+1}} (1 + \varphi_{t+1}) \right] (1 + r_{t+1}) + m_t + a_{t+1}. \end{aligned}$$

Divide the above expression by $p_{t+1} (1 + r_{t+1})$ and rearrange the terms to get

$$c_t + \frac{(1 + \tau) d_{t+1}}{1 + r_{t+1}} + \frac{i_{t+1}}{1 + r_{t+1}} \frac{m_t}{p_{t+1}} = z_t + \frac{a_{t+1}}{p_{t+1} (1 + r_{t+1})}. \tag{11}$$

We next incorporate the Clower cash-in-advance constraint in the intertemporal budget constraint. Substituting for m_t/p_{t+1}, from (7), in the intertemporal budget constraint (11) results in

$$c_t + \frac{(1 + \tau) d_{t+1}}{1 + r_{t+1}} + \frac{i_{t+1}}{1 + r_{t+1}} \left(\gamma^j d_{t+1} - \frac{a_{t+1}}{p_{t+1}} \right) = z_t + \frac{a_{t+1}}{p_{t+1} (1 + r_{t+1})},$$

or, equivalently,

$$c_t + \frac{1 + \tau + \gamma^j i_{t+1}}{1 + r_{t+1}} d_{t+1} = z_t + \frac{a_{t+1}}{p_t}. \tag{12}$$

The problem of a young j-type, who is facing the quintuple $\left(\tau, \theta, a^j_{t+1}, z^j_t, I^j_t\right)$, is to maximize (8) subject to (12). The first-order condition for this problem is

$$\frac{\partial u \left(c_t, d_{t+1}, I_t/w^j_t \right) / \partial d_{t+1}}{\partial u \left(c_t, d_{t+1}, I_t/w^j_t \right) / \partial c_t} = \frac{1 + \tau + \gamma^j i_{t+1}}{1 + r_{t+1}}. \tag{13}$$

Observe that with $\gamma^h \neq \gamma^\ell$, the two types face different effective prices for d_{t+1} (relative to c_t). This is due to the second source of heterogeneity we have postulated. If $\gamma^h = \gamma^\ell$, this latter source of heterogeneity disappears and we will have only the heterogeneity in skills. Condition (13), along with the individual's intertemporal budget constraint (12), yields the following conditional demands for the j-type's first- and second-period consumption,

$$c_t^j = c\left(\frac{1+\tau+\gamma^j i_{t+1}}{1+r_{t+1}}, z_t + \frac{a_{t+1}}{p_t}, \frac{I_t}{w_t^j}\right), \tag{14}$$

$$d_{t+1}^j = d\left(\frac{1+\tau+\gamma^j i_{t+1}}{1+r_{t+1}}, z_t + \frac{a_{t+1}}{p_t}, \frac{I_t}{w_t^j}\right). \tag{15}$$

We summarize our discussion thus far regarding the determination of the temporal equilibrium of this economy as,

Proposition 1 *Consider an overlapping-generations model à la Samuelson (1958) with money wherein money holdings are rationalized by a version of the Clower cash-in-advance constraint. There are two types of agents: One type is skilled and connected, denoted by h; the other type is unskilled and less connected, denoted by ℓ. Both types grow at a constant rate so that the proportion of each type in the total population remains constant over time. Let a young j-type individual face, at time t, the quintuple $\left(\tau, \theta, a_{t+1}^j, z_t^j, I_t^j\right)$, where τ is the tax rate on second-period consumption, θ is the money growth (or contraction) rate, a_{t+1}^j is the j-type's allotment of money injection (or tax withdrawal) to be given in the following period, z_t^j is the j-type's after-tax income, and I_t^j is the j-type's before-tax income; $j = h, \ell$. Under the perfect foresight assumption, the period by period equilibrium of this economy is characterized by equations (1)–(3), (7), and (14)–(15), where the last three equations hold for both $j = h, \ell$.*

2.4 Mechanism designer

It remains for us to specify how the mechanism designer chooses $\left(\tau, \theta, a_{t+1}^j, z_t^j, I_t^j\right)$. This will complete the characterization of the set of (constrained) Pareto-efficient allocations in every period. To simplify notation, introduce

$$q_{t+1}^{j} \equiv \frac{1+\tau+\gamma^{j} i_{t+1}}{1+r_{t+1}}. \tag{16}$$

Substituting these values in (14)–(15), we have

$$\begin{aligned} c_t^j &= c\left(q_{t+1}^j, z_t + \frac{a_{t+1}}{p_t}, \frac{I_t}{w_t^j}\right), \\ d_{t+1}^j &= d\left(q_{t+1}^j, z_t + \frac{a_{t+1}}{p_t}, \frac{I_t}{w_t^j}\right). \end{aligned}$$

Next, substituting the values of c_t^j and d_{t+1}^j in the young j-type's utility function (8), yields his conditional indirect utility function,

$$\begin{aligned} & v\left(q_{t+1}^j, z_t + \frac{a_{t+1}}{p_t}, \frac{I_t}{w_t^j}\right) \equiv \\ & u\left(c\left(q_{t+1}^j, z_t + \frac{a_{t+1}}{p_t}, \frac{I_t}{w_t^j}\right), d\left(q_{t+1}^j, z_t + \frac{a_{t+1}}{p_t}, \frac{I_t}{w_t^j}\right), \frac{I_t}{w_t^j}\right). \end{aligned} \tag{17}$$

To write the incentive-compatibility constraints, we should also know what fraction of his second-period consumption expenditures a j-type who may want to report his type as k, the so-called "mimicker" (or jk agent), must finance through cash balances that he saves in the first period. This fraction may depend on the individual's type as well as the income he earns (when mimicking the other type). Denote this fraction by γ^{jk} for a j-type who mimics a k-type, j and $k = h, \ell$, and corresponding to this introduce

$$q_{t+1}^{jk} \equiv \frac{1+\tau+\gamma^{jk} i_{t+1}}{1+r_{t+1}}. \tag{18}$$

With q_{t+1}^j and q_{t+1}^{jk} given by (16) and (18), the mechanism designer maximizes

$$\sum_{j=\ell,h} \delta^j v\left(q_{t+1}^j, z_t^j + \frac{a_{t+1}^j}{p_t}, \frac{I_t^j}{w_t^j}\right),$$

with respect to $\tau, \theta, a_{t+1}^j, z_t^j$ and I_t^j; subject to the government's budget constraint,

$$n_t^h\left(I_t^h - z_t^h\right) + n_t^\ell\left(I_t^\ell - z_t^\ell\right) + \frac{\tau}{1+r}\left(n_t^h d_{t+1}^h + n_t^\ell d_{t+1}^\ell\right) \geq N_t \bar{R}, \tag{19}$$

the money injection relationship (4), and the self-selection constraints

$$v\left(q^h_{t+1}, z^h_t + \frac{a^h_{t+1}}{p_t}, \frac{I^h_t}{w^h_t}\right) \;\geq\; v\left(q^{h\ell}_{t+1}, z^\ell_t + \frac{a^\ell_{t+1}}{p_t}, \frac{I^\ell_t}{w^h_t}\right), \tag{20}$$

$$v\left(q^\ell_{t+1}, z^\ell_t + \frac{a^\ell_{t+1}}{p_t}, \frac{I^\ell_t}{w^\ell_t}\right) \;\geq\; v\left(q^{\ell h}_{t+1}, z^h_t + \frac{a^h_{t+1}}{p_t}, \frac{I^h_t}{w^\ell_t}\right), \tag{21}$$

where δ^j's are positive constants with the normalization $\sum_{j=\ell,h} \delta^j = 1$, and $\bar{R}$ is an exogenous per-capita revenue requirement. Observe that (19) represents a generational budget constraint as opposed to a per-period budget constraint. We will discuss the solution to this problem, and the properties of the solution, after it reaches its steady-state equilibrium (which we assume exists).

2.5 Some useful expressions

For future reference, define the "real cash balances" that a j-type holds, x^j_t, and the average real cash balances, $\overline{x}_t$, by

$$x^j_t \equiv \frac{m^j_t}{p_{t+1}}, \tag{22}$$

$$\overline{x}_t \equiv \pi^h x^h_t + \pi^\ell x^\ell_t. \tag{23}$$

This allows us, using equation (5), to find the following relationship between x^j_{t+1} and x^j_t,

$$x^j_{t+1} = \frac{1+\theta}{1+g}\frac{x^j_t}{1+\varphi_{t+2}}. \tag{24}$$

Additionally, substituting x^j_t for m^j_t/p_{t+1} in equation (7) yields,

$$x^j_t = \gamma^j d^j_{t+1} - \frac{a^j_{t+1}}{p_t}\frac{1+r_{t+1}}{1+i_{t+1}}. \tag{25}$$

Finally, substituting for M_t from equation (1) into (4) and dividing by $N_t p_t$,

$$\begin{aligned}\pi^h \frac{a^h_{t+1}}{p_t} + \pi^\ell \frac{a^\ell_{t+1}}{p_t} &= \theta\left(\pi^h \frac{m^h_t}{p_{t+1}} + \pi^\ell \frac{m^\ell_t}{p_{t+1}}\right)\frac{p_{t+1}}{p_t} \\ &= \theta\left(\pi^h x^h_t + \pi^\ell x^\ell_t\right)\frac{1+i_{t+1}}{1+r_{t+1}}.\end{aligned}$$

Next, substituting for x_t^j from (25) into above, rearranging the terms and simplifying allows us to rewrite the money injection relationship as

$$\pi^h \frac{a_{t+1}^h}{p_t} + \pi^\ell \frac{a_{t+1}^\ell}{p_t} = \frac{\theta}{1+\theta}\left(\pi^h \gamma^h d_{t+1}^h + \pi^\ell \gamma^\ell d_{t+1}^\ell\right) \frac{1+i_{t+1}}{1+r_{t+1}}. \tag{26}$$

3 Steady state

Denote the steady-state value of the real interest rate by r; this is the fixed rate of return of the storage technology. To derive the corresponding nominal interest rate, observe that in the steady-state, holdings of real cash balances remain constant over time: $x_{t+1}^j = x_t^j \equiv x^j$. This relationship implies, through equation (24), that

$$1+\varphi = \frac{1+\theta}{1+g}.$$

It then follows, from the steady-state version of equation (2), that

$$1+i = \frac{1+r}{1+g}\left(1+\theta\right). \tag{27}$$

Given r and i, the intertemporal price faced by the j-type is determined according to

$$q^j \equiv \frac{1+\tau+\gamma^j i}{1+r}. \tag{28}$$

In steady state, the mechanism designer assigns $I_{t+1}^j = I_t^j, \equiv I^j, z_{t+1}^j = z_t^j \equiv z^j$, and $a_{t+2}^j/p_{t+1} = a_{t+1}^j/p_t \equiv b^j$; $j = h, \ell$. The consumption levels too will then remain constant over time: $c_{t+1}^j = c_t^j \equiv c^j, d_{t+1}^j = d_t^j \equiv d^j$. For ease in notation, introduce

$$y^j \equiv z^j + b^j, \tag{29}$$

to denote the j-type's aggregate disposable income. The steady-state versions of the demand equations for c_t^j and d_{t+1}^j then give us,

$$c^j \equiv c\left(q^j, y^j, \frac{I^j}{w^j}\right), \tag{30}$$

$$d^j \equiv d\left(q^j, y^j, \frac{I^j}{w^j}\right). \tag{31}$$

Similarly, the steady-state value of real cash balances is determined through equation (25) as

$$\begin{aligned} x^j &= \gamma^j d^j - b^j \frac{1+r}{1+i}, \\ &= \gamma^j d^j - b^j \frac{1+g}{1+\theta}. \end{aligned} \tag{32}$$

Other equations of interest are the steady-state versions of the young j-type's intertemporal budget constraint (12) and his conditional indirect utility function (17). These are given by

$$c^j + q^j d^j = y^j, \tag{33}$$

$$v^j = v\left(q^j, y^j, \frac{I^j}{w^j}\right). \tag{34}$$

To derive the steady-state version of the government's budget constraint, divide equation (19) by N_t to write

$$\pi^h\left(I^h - z^h\right) + \pi^\ell\left(I^\ell - z^\ell\right) + \frac{\tau}{1+r}\sum_{j=\ell,h}\pi^j d^j \geq \bar{R}. \tag{35}$$

Additionally, using (27), we can write the steady-state version of the money injection relationship (26) as

$$\pi^h b^h + \pi^\ell b^\ell = \frac{\theta}{1+g}\left(\gamma^h \pi^h d^h + \gamma^\ell \pi^\ell d^\ell\right). \tag{36}$$

Finally, write the mimickers' demands for c and d, and their conditional indirect utility functions. Denoting the steady-state value of q_{t+1}^{jk} by

$$q^{jk} = \frac{1+\tau+\gamma^{jk} i}{1+r}, \tag{37}$$

one can then write

$$c^{jk} = c\left(q^{jk}, y^k, \frac{I^k}{w^j}\right), \tag{38}$$

$$d^{jk} = d\left(q^{jk}, y^k, \frac{I^k}{w^j}\right), \tag{39}$$

$$v^{jk} = v\left(q^{jk}, y^k, \frac{I^k}{w^j}\right). \tag{40}$$

We have,

Proposition 2 *Consider the overlapping-generations model of Proposition 1. Assuming that the model has a steady-state equilibrium, it is characterized by equations (27)–(32). Secondly, let v^j and v^{jk}, defined by equations (34) and (40), denote the conditional indirect utility function of the young j-type and jk-type agents; $j = h, \ell$. Let δ^j's be positive constants with the normalization $\sum_{j=\ell,h} \delta^j = 1$. The constrained Pareto-efficient allocations are described by the maximization of $\sum_{j=\ell,h} \delta^j v^j$ with respect to τ, θ, b^j, z^j and I^j; subject to the government's budget constraint (35), the money injection constraint (36), and the self-selection constraints $v^h \geq v^{h\ell}$ and $v^\ell \geq v^{\ell h}$.*

4 Monetary distribution rule

We now prove that the existence of a general income tax schedule makes monetary distribution rule impotent. Consider, starting from any initial values for b^h and b^ℓ, a change in money disbursements to the h-type and the ℓ-type equal to db^h and db^ℓ. Simultaneously, change z^j according to $dz^j = -db^j$. Now, with $y^j = z^j + b^j$, $dy^j = 0$, and $\left(q^j, y^j, I^j\right)$, $\left(q^{jk}, y^k, I^k\right)$ remain intact. Hence the utility of all agents in the economy including the mimicker, the jk agent, remain the same. As a result, the incentive compatibility constraints continue to be satisfied.

Second, with $\left(q^j, y^j, I^j\right)$ remaining unchanged, the j-type's demand for d does not change either. Consequently, the changes in b^j imply, from the money injection constraint (36), that

$$\begin{aligned}\pi^h db^h + \pi^\ell db^\ell &= \frac{\theta}{1+g}\left(\pi^\ell \gamma^\ell dd^\ell + \pi^h \gamma^h dd^h\right) \\ &= 0.\end{aligned} \tag{41}$$

Third, with d^j not changing, the only change in the government's revenue requirement comes from the changes in z^j. Hence, from (35) and (41),

$$\begin{aligned}dR &= -\left(\pi^h dz^h + \pi^\ell dz^\ell\right) \\ &= \pi^h db^h + \pi^\ell db^\ell = 0.\end{aligned}$$

We thus have shown that the considered changes satisfy all the constraints that the economy faces but leaves every agent as well off as he was before.

The import of all this is that the redistributive effects of increasing the monetary disbursements to one type of agents and reducing them to the other, such that the aggregate money injection to the economy remains the same, can always be offset by changes in the individuals' income tax payments. The welfare of all agents remain unaffected. This holds true whether the initial equilibrium, corresponding to the initial values of b^h and b^ℓ, was optimal or not.

It is important to point out that this result does not contradict Williamson's (2008) who finds the monetary expansion rule does matter. Nor is the two different results due to the fact that in Williamson's setup, there is no fiscal authority to *try* to undo what the monetary authority does. The underlying factor is the distinction he makes between the connected and unconnected agents in terms of their access to financial institutions. The impact of this source of heterogeneity does not show up in b^j. In our model, this distinction manifests itself through different γ's that the two types face with respect to their cash-in-advance constraints. This, in turn, manifests itself through q^j and not b^j.

This is summarized as

Proposition 3 *Consider the steady-state equilibrium of our overlapping-generations model with cash-in-advance constraint and with heterogeneous agents. For a given monetary rate of growth, the fiscal authority can offset the redistributive effects of who gets the extra money (or loses the money that is withdrawn from the economy), by adjusting the individuals' income tax payments. All agents will continue to enjoy the same level of welfare.*

5 Monetary growth rate

Consider now, starting from any initial value for θ, a change in the monetary growth rate equal to $d\theta$. To determine how this changes q^j, substitute for i from (27) in (28) to get

$$q^j = \frac{1}{1+r} + \gamma^j \left(\frac{1}{1+g} - \frac{1}{1+r} \right) + \frac{\tau}{1+r} + \frac{\gamma^j \theta}{1+g}. \tag{42}$$

It follows from (42) that

$$dq^j \equiv \frac{\gamma^j}{1+g} d\theta.$$

It is clear from the above expression that a change in θ changes q^j differently for individuals of different types. As long as the government has to tax future goods at the same rate for everyone, it will be impossible to offset the effect of a change in θ with a change in τ. Consequently, this aspect of monetary policy cannot be neutralized with fiscal policy.[13]

5.1 Skills as the sole source of heterogeneity

With $\gamma^j = \gamma$, from (42), q^j simplifies to

$$q = \frac{1}{1+r} + \gamma \left(\frac{1}{1+g} - \frac{1}{1+r} \right) + \frac{\tau}{1+r} + \frac{\gamma \theta}{1+g}. \tag{43}$$

[13]This discussions alerts us to the fact that if the fiscal authority could tax consumption goods *at different rates for different individuals*, it would be able to offset the change in q^j to both individual types. Under this assumption, the fiscal authority has enough information to set the commodity tax rates differently for different agents. This information structure is patently unrealistic. We thus investigate its implications only in an appendix; see Appendix F.

To check the implications of this case, consider now, starting from any initial values for τ and θ, a change in the growth rate of money equal to $d\theta$ while offsetting it with a corresponding change in τ that keeps q constant. It follows from (43) that one has to set

$$d\tau = \frac{1+r}{1+g}\left(-\gamma d\theta\right), \tag{44}$$

in order to have $dq = 0$.

Next observe that the change in θ induces a change in b^j as well. As in the exercise of Section 4, let the fiscal authority also change z^j according to $dz^j = -db^j$. This change ensures that $dy^j = dz^j + db^j = 0$. With $dy^j = dq^j = 0$ and no change in I^j, the instituted changes leave the utility of the h-types and the ℓ-types intact. Observe also that the utility of potential mimickers, the jk-agents, remain unaffected as they continue to face the same price and income vector $\left(q, y^k, I^k\right)$. Consequently, the incentive compatibility constraints continue to be satisfied. Thus, if the considered changes do not violate the government's budget constraint, they constitute a feasible change that leaves every agent just as well off as initially.

To check this, observe first that with $\left(q, y^j, I^j\right)$ remaining unchanged, the j-type's demand for d does not change either. With $dd^j = 0$, the change in the government's net tax revenue is, from (35),

$$dR = -\left(\pi^h dz^h + \pi^\ell dz^\ell\right) + \frac{d\tau}{1+r}\sum_{j=\ell,h}\pi^j d^j.$$

Substituting $-db^j$ for dz^j and the value of $d\tau$ from (44) in above, we get

$$dR = \pi^h db^h + \pi^\ell db^\ell - \frac{\gamma d\theta}{1+g}\sum_{j=\ell,h}\pi^j d^j. \tag{45}$$

Now note that the changes in θ and b^j must satisfy the money injection constraint equation (36). Given that $dd^j = 0$, we have

$$\sum_{j=\ell,h}\pi^j db^j = \frac{\gamma\sum_{j=\ell,h}\pi^j d^j}{1+g}d\theta. \tag{46}$$

Substituting from (46) into (45) results in $dR = 0$.

This exercise tells us that, for every feasible rate of money injection, the fiscal authority can set a tax rate on second-period consumption, and adjust the income tax rates of the agents, in such a way as to keep the welfare of everybody intact. Observe that the described reform applies to any initial values of τ and θ; that is, for any initial value of q. This includes the case where the society's welfare was initially maximal. An implication of this is that the optimal monetary growth rate is not unique; a continuum of values satisfies it.

The results of this section are summarized in the following Proposition.

Proposition 4 *Consider the steady-state equilibrium of our overlapping-generations model with cash-in-advance constraint and with heterogeneous agents.*

(i) A change in monetary growth rate changes the relative price of future to present consumption differently for different individuals. The fiscal authority cannot neutralize the effects of such a change in monetary policy.

(ii) If the only source of heterogeneity is skill levels, $\gamma^h = \gamma^\ell = \gamma$ and the fiscal authority is able to neutralize the effects of a change in the rate of monetary growth. Under this circumstance, the optimal monetary growth rate is not unique. Social welfare is maximized by a continuum of values for the monetary growth rate, θ, and the tax on the second-period consumption, τ (coupled with supporting income tax rates).

6 Second-best characterization

In formulating the second-best optimization problem, we follow the common practice in the optimal income tax literature and ignore the "upward" incentive constraint, $v^\ell \geq v^{\ell h}$; assuming that it is automatically satisfied. Thus, the only possible binding constraint will be that of the high-skilled agents mimicking low-skilled agents. Intuitively, this implies that we are concerned only with the realistic case of redistribution from the high-skilled to low-skilled agents.[14]

Denote the Lagrangian expression associated with the government's problem described in Section 3 by $\mathcal{L}$, the Lagrangian multipliers associated with the government's budget constraint (35) by μ, with the money injection constraint (36) by η, and with the self-selection constraint $v^h \geq v^{h\ell}$ by λ.[15] One can then write

$$\begin{aligned}\mathcal{L} &= \sum_{j=\ell,h} \delta^j v^j + \lambda\left(v^h - v^{h\ell}\right) + \eta\left[\sum_{j=\ell,h} \pi^j\left(b^j - \frac{\theta}{1+g}\gamma^j d^j\right)\right] \\ &\quad +\mu\left[\sum_{j=\ell,h} \pi^j\left(I^j - z^j + \frac{\tau}{1+r}d^j\right) - \bar{R}\right], \end{aligned} \tag{47}$$

or equivalently, given that from (27) we have that $\frac{\theta}{1+g} = \frac{1}{1+r}\left(\frac{g-r}{1+g} + i\right)$,

$$\begin{aligned}\mathcal{L} &= \sum_{j=\ell,h} \delta^j v^j + \lambda\left(v^h - v^{h\ell}\right) + \eta\left\{\sum_{j=\ell,h} \pi^j\left[b^j - \frac{1}{1+r}\left(\frac{g-r}{1+g} + i\right)\gamma^j d^j\right]\right\} \\ &\quad +\mu\left[\sum_{j=\ell,h} \pi^j\left(I^j - z^j + \frac{\tau}{1+r}d^j\right) - \bar{R}\right]. \end{aligned} \tag{48}$$

Let $\widetilde{d^j}$ denote the j-type's compensated (Hicksian) demand for d. Manipulating the first-order conditions of this problem, we prove in Appendix A,

[14] Given the perfect correlation between skills and connectedness, the properties of our setting with two sources of heterogeneity reduces to that of a two-group model à la Stiglitz (1982). In particular, the single crossing property is satisfied in the usual manner and there will at most be one binding self-selection constraint.

[15] This formulation considers the steady-state utilities only. This is not to suggest that the welfare of individuals on the transition path does not matter. It is just that considering them does not change the points addressed in this paper and makes the presentation much more cumbersome. One can also rationalize this approach by assuming a Millian social welfare function over undiscounted average utilities of all present and future generations.

$$
\begin{aligned}
\tau &= \frac{(1+r)\,\lambda\alpha^{h\ell}}{\Delta\mu}\frac{\pi^{\ell}\gamma^{\ell}\partial\widetilde{d}^{\ell}/\partial q^{\ell}}{1-d^{\ell}\frac{i}{1+r}\frac{\partial\gamma^{\ell}}{\partial y^{\ell}}}\left(\gamma^{h\ell}-\gamma^{\ell}\right)d^{h\ell} \\
&+\frac{(1+r)\,\lambda\alpha^{h\ell}}{\Delta\mu}\frac{\pi^{h}\gamma^{h}\partial\widetilde{d}^{h}/\partial q^{h}}{1-d^{h}\frac{i}{1+r}\frac{\partial\gamma^{h}}{\partial y^{h}}}\left[\left(\gamma^{h\ell}-\gamma^{h}\right)d^{h\ell}-\left(\gamma^{\ell}-\gamma^{h}\right)\frac{1-d^{h\ell}\frac{i}{1+r}\frac{\partial\gamma^{h\ell}}{\partial y^{\ell}}}{1-d^{\ell}\frac{i}{1+r}\frac{\partial\gamma^{\ell}}{\partial y^{\ell}}}d^{\ell}\right] \\
&+\frac{g-r}{1+g}\frac{\frac{\partial\gamma^{\ell}}{\partial y^{\ell}}\left(d^{\ell}\right)^{2}\gamma^{h}\frac{\partial\widetilde{d}^{h}}{\partial q^{h}}-\frac{\partial\gamma^{h}}{\partial y^{h}}\left(d^{h}\right)^{2}\gamma^{\ell}\frac{\partial\widetilde{d}^{\ell}}{\partial q^{\ell}}}{\left(\gamma^{\ell}-\gamma^{h}\right)\frac{\partial\widetilde{d}^{h}}{\partial q^{h}}\frac{\partial\widetilde{d}^{\ell}}{\partial q^{\ell}}}, \qquad (49)
\end{aligned}
$$

$$
\begin{aligned}
i &= \frac{(1+r)\,\lambda\alpha^{h\ell}}{\Delta\mu}\frac{\pi^{\ell}\partial\widetilde{d}^{\ell}/\partial q^{\ell}}{1-d^{\ell}\frac{i}{1+r}\frac{\partial\gamma^{\ell}}{\partial y^{\ell}}}\left(\gamma^{\ell}-\gamma^{h\ell}\right)d^{h\ell} \\
&+\frac{(1+r)\,\lambda\alpha^{h\ell}}{\Delta\mu}\frac{\pi^{h}\partial\widetilde{d}^{h}/\partial q^{h}}{1-d^{h}\frac{i}{1+r}\frac{\partial\gamma^{h}}{\partial y^{h}}}\left[\left(\gamma^{\ell}-\gamma^{h}\right)\frac{1-d^{h\ell}\frac{i}{1+r}\frac{\partial\gamma^{h\ell}}{\partial y^{\ell}}}{1-d^{\ell}\frac{i}{1+r}\frac{\partial\gamma^{\ell}}{\partial y^{\ell}}}d^{\ell}-\left(\gamma^{h\ell}-\gamma^{h}\right)d^{h\ell}\right] \\
&+\frac{g-r}{1+g}\left[\left(\frac{\frac{\partial\gamma^{h}}{\partial y^{h}}\left(d^{h}\right)^{2}}{\frac{\partial\widetilde{d}^{h}}{\partial q^{h}}}-\frac{\frac{\partial\gamma^{\ell}}{\partial y^{\ell}}\left(d^{\ell}\right)^{2}}{\frac{\partial\widetilde{d}^{\ell}}{\partial q^{\ell}}}\right)\frac{1}{\gamma^{\ell}-\gamma^{h}}-1\right], \qquad (50)
\end{aligned}
$$

where $\Delta \equiv \pi^{\ell}\pi^{h}\frac{\partial\widetilde{d}^{\ell}}{\partial q^{\ell}}\frac{\partial\widetilde{d}^{h}}{\partial q^{h}}\frac{\left(\gamma^{\ell}-\gamma^{h}\right)^{2}}{\left(1-d^{h}\frac{i}{1+r}\frac{\partial\gamma^{h}}{\partial y^{h}}\right)\left(1-d^{\ell}\frac{i}{1+r}\frac{\partial\gamma^{\ell}}{\partial y^{\ell}}\right)} > 0$, and $\alpha^{h\ell} > 0$ denotes the mimicker's marginal utility of income.

Postponing the analysis of how the desirability of abiding by the Friedman rule is affected when $g \neq r$, let's start assuming that $g = r$. Then, from eq. (50) we can see that the Friedman rule is optimal, either as an interior or as a corner solution, when:

$$
d^{h\ell}\sum_{j=\ell,h}\pi^{j}\frac{\partial\widetilde{d}^{j}}{\partial q^{j}}\frac{1}{1-d^{j}\frac{i}{1+r}\frac{\partial\gamma^{j}}{\partial y^{j}}}\left(\gamma^{h\ell}-\gamma^{j}\right) \geq d^{\ell}\frac{1-d^{h\ell}\frac{i}{1+r}\frac{\partial\gamma^{h\ell}}{\partial y^{\ell}}}{1-d^{\ell}\frac{i}{1+r}\frac{\partial\gamma^{\ell}}{\partial y^{\ell}}}\left(\gamma^{\ell}-\gamma^{h}\right)\pi^{h}\frac{\partial\widetilde{d}^{h}}{\partial q^{h}}\frac{1}{1-d^{h}\frac{i}{1+r}\frac{\partial\gamma^{h}}{\partial y^{h}}}. \qquad (51)
$$

With $\gamma(\cdot)$ being decreasing in skill levels and incomes, $\gamma^{h} < \gamma^{h\ell} < \gamma^{\ell}$, and the right hand side of (51) is negative. Thus, a sufficient condition for (51) to be satisfied is that $\sum_{j=\ell,h}\frac{\pi^{j}\partial\widetilde{d}^{j}/\partial q^{j}}{1-d^{j}\frac{i}{1+r}\frac{\partial\gamma^{j}}{\partial y^{j}}}\left(\gamma^{h\ell}-\gamma^{j}\right) \geq 0$, i.e.:

$$\underbrace{\frac{\pi^{\ell}\partial\widetilde{d}^{\ell}/\partial q^{\ell}}{1-d^{\ell}\frac{i}{1+r}\frac{\partial\gamma^{\ell}}{\partial y^{\ell}}}\left(\gamma^{\ell}-\gamma^{h\ell}\right)}_{<0} \leq \underbrace{\frac{\pi^{h}\partial\widetilde{d}^{h}/\partial q^{h}}{1-d^{h}\frac{i}{1+r}\frac{\partial\gamma^{h}}{\partial y^{h}}}\left(\gamma^{h\ell}-\gamma^{h}\right)}_{<0},$$

or, equivalently:

$$\gamma^{\ell}-\gamma^{h\ell} \geq \left(\gamma^{h\ell}-\gamma^{h}\right)\frac{\pi^{h}}{\pi^{\ell}}\frac{\frac{\partial\widetilde{d}^{h}}{\partial q^{h}}}{\frac{\partial\widetilde{d}^{\ell}}{\partial q^{\ell}}}\frac{1-d^{\ell}\frac{i}{1+r}\frac{\partial\gamma^{\ell}}{\partial y^{\ell}}}{1-d^{h}\frac{i}{1+r}\frac{\partial\gamma^{h}}{\partial y^{h}}}. \tag{52}$$

The above inequality is more likely to be satisfied as $\gamma^{h\ell}$ gets closer to γ^{h} widening the difference between γ^{ℓ} and $\gamma^{h\ell}$. In the special case when $\gamma(\cdot)$ depends (negatively) only on skills, we have $\gamma^{\ell} > \gamma^{h\ell} = \gamma^{h}$ and the Friedman rule is optimal. Another condition which strengthens the case for the optimality of the Friedman rule is given by a small value for the ratio π^{h}/π^{ℓ}. At the limit, when the proportion of high-skilled agents becomes negligible, π^{h}/π^{ℓ} approaches zero and the Friedman rule is once again optimal. These results are summarized in the following Proposition.

Proposition 5 *Assuming that $g = r$, in the steady-state equilibrium of our OLG model with cash-in-advance constraints and heterogeneous agents, abiding by the Friedman rule is part of an optimal policy irrespective of the assumptions on the individuals' preferences when $\sum_{j=\ell,h}\frac{\pi^{j}\partial\widetilde{d}^{j}/\partial q^{j}}{1-d^{j}\frac{i}{1+r}\frac{\partial\gamma^{j}}{\partial y^{j}}}\left(\gamma^{h\ell}-\gamma^{j}\right) \geq 0$. This happens for instance when $\gamma(\cdot)$ depends (negatively) only on skills or when π^{h} is very small.*

If instead $\sum_{j=\ell,h}\frac{\pi^{j}\partial\widetilde{d}^{j}/\partial q^{j}}{1-d^{j}\frac{i}{1+r}\frac{\partial\gamma^{j}}{\partial y^{j}}}\left(\gamma^{h\ell}-\gamma^{j}\right) < 0$, condition (51) requires

$$d^{h\ell} \leq d^{\ell}\frac{1-d^{h\ell}\frac{i}{1+r}\frac{\partial\gamma^{h\ell}}{\partial y^{\ell}}}{1-d^{\ell}\frac{i}{1+r}\frac{\partial\gamma^{\ell}}{\partial y^{\ell}}}\frac{\left(\gamma^{\ell}-\gamma^{h}\right)\frac{\pi^{h}\partial\widetilde{d}^{h}/\partial q^{h}}{1-d^{h}\frac{i}{1+r}\frac{\partial\gamma^{h}}{\partial y^{h}}}}{\sum_{j=\ell,h}\frac{\pi^{j}\partial\widetilde{d}^{j}/\partial q^{j}}{1-d^{j}\frac{i}{1+r}\frac{\partial\gamma^{j}}{\partial y^{j}}}\left(\gamma^{h\ell}-\gamma^{j}\right)},$$

or, equivalently:

$$d^{h\ell} \le d^{\ell} \frac{1 - d^{h\ell} \frac{i}{1+r} \frac{\partial \gamma^{h\ell}}{\partial y^{\ell}}}{1 - d^{\ell} \frac{i}{1+r} \frac{\partial \gamma^{\ell}}{\partial y^{\ell}}} \frac{\gamma^{\ell} - \gamma^{h}}{\gamma^{h\ell} - \gamma^{h} + \left(\gamma^{h\ell} - \gamma^{\ell}\right) \frac{\pi^{\ell}}{\pi^{h}} \frac{\frac{\partial \tilde{d}^{\ell}}{\partial q^{\ell}}}{\frac{\partial \tilde{d}^{h}}{\partial q^{h}}} \frac{1 - d^{h} \frac{i}{1+r} \frac{\partial \gamma^{h}}{\partial y^{h}}}{1 - d^{\ell} \frac{i}{1+r} \frac{\partial \gamma^{\ell}}{\partial y^{\ell}}}}. \tag{53}$$

Consider the special case when $\gamma(\cdot)$ depends (negatively) only on income, either gross income or aggregate disposable income, so that $\gamma^{\ell} = \gamma^{h\ell} > \gamma^{h}$ and $\partial\gamma^{h\ell}/\partial y^{\ell} = \partial\gamma^{\ell}/\partial y^{\ell}$.[16] Then (53) boils down to $d^{h\ell} \le d^{\ell}$. From (28) and (37) we can see that when $\gamma^{\ell} = \gamma^{h\ell}$ a low-skilled agent and a high-skilled mimicker face the same intertemporal price, $q^{\ell} = q^{h\ell}$. Then, a necessary and sufficient condition for $d^{h\ell} \le d^{\ell}$ is that labor supply and second-period consumption d are not Hicksian substitutes (this is because high-skilled workers who plan to pass themselves out as low-skilled workers, earning I^{ℓ} and paying $I^{\ell} - z^{\ell}$ in taxes, work less than low-skilled workers). This result is stated in the following Proposition.

Proposition 6 *Assuming that $g = r$, when $\sum_{j=\ell,h} \frac{\pi^{j} \partial \tilde{d}^{j}/\partial q^{j}}{1 - d^{j} \frac{i}{1+r} \frac{\partial \gamma^{j}}{\partial y^{j}}} \left(\gamma^{h\ell} - \gamma^{j}\right) < 0$ and $\gamma(\cdot)$ depends (negatively) only on income, a necessary and sufficient condition for the optimality of the Friedman rule is that labor supply and second period consumption are not Hicksian substitutes.*

In the general case when $\gamma(\cdot)$ depends (negatively) both on income and skills, so that $\gamma^{\ell} > \gamma^{h\ell} > \gamma^{h}$, things get more complicated and results more ambiguous. However, some progress can be made by assuming that the function $\gamma(\cdot)$ is quasi-linear in income. If this is the case, and if the relevant income variable is represented by gross income, one can show that a weaker condition is sufficient to guarantee that (53) holds and that the Friedman rule is optimal, provided that the own-price elasticity of demand for second period consumption is not too large. In particular, one can show that (see Appendix B for details):

[16] When $\gamma(\cdot)$ depends only on gross income I, we have that both $\partial\gamma^{h\ell}/\partial y^{\ell}$ and $\partial\gamma^{\ell}/\partial y^{\ell}$ are equal to zero. When $\gamma(\cdot)$ depends only on aggregate disposable income y, we have that both $\partial\gamma^{h\ell}/\partial y^{\ell}$ and $\partial\gamma^{\ell}/\partial y^{\ell}$ take the same negative value. Thus, in either case it is true that $\partial\gamma^{h\ell}/\partial y^{\ell} = \partial\gamma^{\ell}/\partial y^{\ell}$.

Proposition 7 *Assuming that* $g = r$*, when* $\sum_{j=\ell,h} \pi^j \frac{\partial \widetilde{d}^j}{\partial q^j} \left(\gamma^{h\ell} - \gamma^j\right) < 0$*,* $\gamma(\cdot) = \gamma(w, I)$ *with* $\gamma(\cdot)$ *quasi-linear in gross income, and* $\mid \varepsilon_{d,q} \mid$ *is small enough, Hicksian non-substitutability between labor supply and second period consumption is a sufficient but no longer necessary condition for the Friedman rule to be optimal; the Friedman rule can be optimal even under a moderate degree of substitutability between* L *and* d*. When instead* $\mid \varepsilon^{h\ell}_{d,q} \mid$ *is sufficiently large, Hicksian complementarity between labor supply and second period consumption is a necessary but no longer sufficient condition for the Friedman rule to be optimal; the complementarity between* L *and* d *must be sufficiently strong for the Friedman rule to be optimal.*

Maintaining the assumption that $\gamma(\cdot)$ is quasi-linear in income but assuming that the relevant income variable is aggregate disposable income, so that $\gamma(\cdot) = \gamma(w, y)$, a slightly different result, stated in the following Proposition, can be derived (see Appendix C for details).

Proposition 8 *Assuming that* $g = r$*, when* $\sum_{j=\ell,h} \frac{\pi^j \partial \widetilde{d}^j / \partial q^j}{1 - d^j \frac{i}{1+r} \frac{\partial \gamma^j}{\partial y^j}} \left(\gamma^{h\ell} - \gamma^j\right) < 0$*,* $\gamma(\cdot) = \gamma(w, y)$ *with* $\gamma(\cdot)$ *quasi-linear in aggregate disposable income, and* $\mid \partial\gamma/\partial y \mid$ *is large, Hicksian non-substitutability between labor supply and second period consumption is a sufficient but no longer necessary condition for the Friedman rule to be optimal; the Friedman rule can be optimal even under a moderate degree of substitutability between* L *and* d*. When instead* $\mid \partial\gamma/\partial y \mid$ *is small, a large absolute value of* $\mid \varepsilon_{d,q} \mid$ *implies that Hicksian complementarity between labor supply and second period consumption is a necessary but no longer sufficient condition for the Friedman rule to be optimal; the complementarity between* L *and* d *must be sufficiently strong for the Friedman rule to be optimal.*

Let's now look at how the desirability to abide by the Friedman rule is affected by a discrepancy between g and r. To address the analysis in a gradual manner, consider first the case when $\gamma(\cdot)$ does not depend on aggregate disposable income y. This can happen either when $\gamma(\cdot)$ depends only on skills, viz. $\gamma(\cdot) = \gamma(w)$, or when it depends both on skills and before-tax labor income, viz. $\gamma(\cdot) = \gamma(w, I)$. In both cases, since $\partial\gamma^{\ell}/\partial y^{\ell} = \partial\gamma^{h}/\partial y^{h} = \partial\gamma^{h\ell}/\partial y^{\ell} = 0$, (50) simplifies to:

$$i = \frac{(1+r)\,\lambda\alpha^{h\ell}}{\Delta\mu}\left\{\pi^{\ell}\frac{\partial\widetilde{d}^{\ell}}{\partial q^{\ell}}\left(\gamma^{\ell}-\gamma^{h\ell}\right)d^{h\ell}+\pi^{h}\frac{\partial\widetilde{d}^{h}}{\partial q^{h}}\left[\left(\gamma^{\ell}-\gamma^{h}\right)d^{\ell}-\left(\gamma^{h\ell}-\gamma^{h}\right)d^{h\ell}\right]\right\}-\frac{g-r}{1+g}. \tag{54}$$

The last term in eq. (54) reflects the fact that, as one can see from the η-constraint in (48), the planner cares about the overall value of $i + (g-r)/(1+g)$, not just of i. This circumstance implies that, when $r < g$, the optimal value of i is mechanically lowered, making the Friedman rule more likely to be optimal. On the other hand, when $g < r$ there is a force at work pushing in the direction of raising i and deviating from the Friedman rule. We can then state the following Proposition.

Proposition 9 *When $\gamma(\cdot)$ depends only on skills or when it depends on skills and before-tax income, abiding by the Friedman rule is more likely (resp.: less likely) to be optimal when $g > r$ (resp.: $g < r$).*

Things get instead more complicated when the dependence of $\gamma(\cdot)$ is with aggregate disposable income y so that $\gamma(\cdot) = \gamma(w, y)$. In this case $\partial\gamma^{j}/\partial y^{j} < 0$ and the term related to $g - r$ in (50) can no longer be unambiguously signed. To shed light on what generates this ambiguity and to get more insights into the analytical results provided by (49) and (50), note that in our setup both τ and i act as a tax on second-period consumption and can help increase redistribution from the high- to low-ability individuals (beyond what one can do with a general income tax alone). The question is why the two instruments play distinct roles. After all what matters is the wedge between future and present consumption (and not the values of τ and i per-se). To answer this, consider the "effective" tax rate on d faced by a j-type agent. This is given by $t^{j} = q^{j} - 1/(1+r)$, i.e., from (28):

$$t^j = \frac{1+\tau+\gamma^j i}{1+r} - \frac{1}{1+r} = \frac{\tau+\gamma^j i}{1+r}. \tag{55}$$

That we have two different expressions for t^h and t^ℓ explains why one cannot substitute fiscal for monetary policy when creating a wedge between future and present consumption. A change in i affects the two individual types differently (one having γ^h and the other γ^ℓ). This is not the case for τ. It is this feature that makes monetary policy different from fiscal policy—a feature due to the heterogeneity of agents in a dimension different from skills.[17]

Substituting for τ and i from (49) and (50) in (55) and simplifying, we prove in Appendix D,

$$t^h = \frac{\lambda\alpha^{h\ell}\left(1 - d^h \frac{i}{1+r}\frac{\partial\gamma^h}{\partial y^h}\right)}{\mu\pi^h \frac{\partial\widetilde{d}^h}{\partial q^h}} \frac{\gamma^{h\ell}-\gamma^\ell}{\gamma^\ell-\gamma^h} d^{h\ell} - \frac{g-r}{(1+g)(1+r)}\left[\underbrace{\frac{\frac{\partial\gamma^h}{\partial y^h}\left(d^h\right)^2}{\frac{\partial\widetilde{d}^h}{\partial q^h}} + \gamma^h}_{>0}\right], \tag{56}$$

$$\begin{aligned} t^\ell \;=\; & \frac{\lambda\alpha^{h\ell}\left(1 - d^\ell \frac{i}{1+r}\frac{\partial\gamma^\ell}{\partial y^\ell}\right)}{\mu\pi^\ell \frac{\partial\widetilde{d}^\ell}{\partial q^\ell}} \left(\frac{1 - d^{h\ell}\frac{i}{1+r}\frac{\partial\gamma^{h\ell}}{\partial y^\ell}}{1 - d^\ell \frac{i}{1+r}\frac{\partial\gamma^\ell}{\partial y^\ell}} d^\ell - \frac{\gamma^{h\ell}-\gamma^h}{\gamma^\ell-\gamma^h} d^{h\ell}\right) \\ & - \frac{g-r}{(1+g)(1+r)}\left[\underbrace{\frac{\frac{\partial\gamma^\ell}{\partial y^\ell}\left(d^\ell\right)^2}{\frac{\partial\widetilde{d}^\ell}{\partial q^\ell}} + \gamma^\ell}_{>0}\right]. \end{aligned} \tag{57}$$

The first term on the right-hand sides of (56) and (57) reflects the incentive effects of our policy instruments, namely the role of t^h and t^ℓ as instruments to slacken the

[17] Otherwise, with the same value for γ for the two types, there will be one effective tax rate for the two types with i and τ playing identical roles. Under this latter circumstance, as we saw in Subsection 5.1, the choice of τ or i does not matter. The fiscal authority can always neutralize the effect of i through an appropriate choice of τ.

relevant self-selection constraint. With $\gamma(\cdot)$ being decreasing in skill levels and incomes, $\gamma^h < \gamma^{h\ell} < \gamma^\ell$ and the incentive term on t^h is positive. As to the incentive term on t^ℓ, it will certainly be negative if $d^{h\ell} < d^\ell$. This follows because $\left(\gamma^{h\ell} - \gamma^h\right) / \left(\gamma^\ell - \gamma^h\right) < 1$. On the other hand, if $d^{h\ell} > d^\ell$ the sign of the incentive term on t^ℓ is ambiguous.

Assuming $g = r$ in order to abstract for the moment from the last term in (56)-(57), it is of interest to interpret the structure of the incentive terms. To do this, we follow a perturbation approach. Starting with (56), consider the following small tax reform around an interior optimum. Raise i marginally while at the same time lower τ by γ^ℓ in order to leave q^ℓ unchanged; then change z^h properly in order to offset any effect on the utility of high-ability agents coming from the induced variation in q^h. With no effects on the utility of agents of type h and ℓ, the only remaining effects of the reform are on the self-selection constraint, the government's budget constraint and the money-injection constraint. From the Lagrangian (48), we can see that the effect on the self-selection constraint is given by

$$-\lambda dv^{h\ell} = -\lambda \frac{\partial v^{h\ell}}{\partial q^{h\ell}} dq^{h\ell} = \lambda \alpha^{h\ell} d^{h\ell} \frac{d\tau + \gamma^{h\ell}}{1+r} = \lambda \alpha^{h\ell} d^{h\ell} \frac{\gamma^{h\ell} - \gamma^\ell}{1+r}.$$

Since in Appendix A we show that $\mu = -\eta$, the effects on the government's budget constraint and the money-injection constraint can be combined to obtain:

$$\pi^h \left(\mu \frac{\tau}{1+r} \frac{\partial \widetilde{d}^h}{\partial q^h} dq^h - \eta \frac{\gamma^h i}{1+r} \frac{\partial \widetilde{d}^h}{\partial q^h} dq^h \right) = \mu \pi^h \left(\frac{\tau}{1+r} + \frac{i \gamma^h}{1+r} \right) \frac{\partial \widetilde{d}^h}{\partial q^h} \frac{1}{1 - d^h \frac{i}{1+r} \frac{\partial \gamma^h}{\partial y^h}} \frac{\gamma^h - \gamma^\ell}{1+r}.$$

If the pre-reform equilibrium was an optimum, the two effects should exactly offset. This requires:

$$\mu \pi^h \left(\frac{\tau}{1+r} + \frac{\gamma^h i}{1+r} \right) \frac{\partial \widetilde{d}^h}{\partial q^h} \frac{1}{1 - d^h \frac{i}{1+r} \frac{\partial \gamma^h}{\partial y^h}} \left(\gamma^\ell - \gamma^h \right) = \lambda \alpha^{h\ell} d^{h\ell} \left(\gamma^{h\ell} - \gamma^\ell \right). \tag{58}$$

Taking into account that $t^h \equiv \left(\tau + \gamma^h i\right) / (1+r)$ and rearranging terms in (58) gives:

$$t^h = \frac{\lambda\alpha^{h\ell}\left(1 - d^h \frac{i}{1+r}\frac{\partial\gamma^h}{\partial y^h}\right)}{\mu\pi^h \frac{\partial\widetilde{d}^h}{\partial q^h}} \frac{\gamma^{h\ell} - \gamma^\ell}{\gamma^\ell - \gamma^h} d^{h\ell},$$

which rationalizes the incentive term appearing in (56).

A similar perturbation approach can be used to interpret the structure of the incentive term appearing in (57). More precisely, the incentive term appearing in (57) can be rationalized by thinking at the experiment of marginally raising i while at the same time: i) lowering τ by γ^h in order to leave q^h unchanged, and ii) changing z^ℓ properly in order to offset any effect on the utility of low-ability agents coming from the induced variation in q^ℓ. With no effects on the utility of agents of type h and ℓ, the only remaining effects of the reform are on the self-selection constraint, the government's budget constraint and the money-injection constraint. If the pre-reform equilibrium was an optimum these effects should exactly offset, which is what can be shown eq. (57) requires.[18]

The second term on the right hand sides of (56) and (57) reflects a disconnect, which arises when $r \neq g$, between the private and the social intertemporal wedge. While the private wedge is given by (55), from (47)-(48) and the fact that $\mu = -\eta$ at an optimum, the social wedge is given by:

$$\frac{\tau}{1+r} + \frac{\gamma^j\theta}{1+g} = \frac{\tau}{1+r} + \frac{\gamma^j}{1+r}\left(\frac{g-r}{1+g} + i\right). \tag{59}$$

Thus, when $g < r$ the social wedge is smaller than the private wedge and vice versa. This disconnect has two consequences for the optimal private wedge faced by agents. The first is a mechanical adjustment captured by the second term within square brackets in (56)-(57). The second is related to the dependence of $\gamma(\cdot)$ on disposable income and is captured by the first term within square brackets in (56)-(57). This

[18] See Appendix E for details.

term encapsulates the net budget effect for the planner of the reduction in γ^j which accompanies a compensated marginal increase in the private wedge t^j. With $\partial\gamma^j/\partial y^j < 0$, the increase in after-tax income required to offset the negative effect on the agents' utility of a marginal increase in t^j is lower, by an amount proportional to $\left(d^j\right)^2 \frac{\partial t^j}{\partial \gamma^j}\frac{\partial \gamma^j}{\partial y^j} = \left(d^j\right)^2 \frac{i}{1+r}\frac{\partial \gamma^j}{\partial y^j}$, than it would be if $\partial\gamma^j/\partial y^j$ were nil. Besides this positive budget effect for the planner, the induced reduction in γ^j has also a negative budget effect due to the fact that it triggers a reduction in the social wedge (59). The net effect, proportional to $\left(d^j\right)^2 \frac{g-r}{(1+g)(1+r)}\partial\gamma^j/\partial y^j$ and captured by the first term within square brackets in (56)-(57), will be positive whenever the social wedge (59) is smaller than the private wedge (55), i.e. whenever $g < r$.

A crucial thing to notice about (56)-(57) is that they express the optimal effective tax rates on second-period consumption *if* τ and i could both be freely chosen. In reality, however, the choice of i is subject to a non-negativity constraint which implies, together with $\gamma^h < \gamma^\ell$, that the solution provided by (56)-(57) is unfeasible when it prescribes $t^h = \left(\tau + \gamma^h i\right)/(1+r) > t^\ell = \left(\tau + \gamma^\ell i\right)/(1+r)$. Thus, when (56)-(57) prescribe $t^h > t^\ell$, abiding by the Friedman rule is again part of an optimal policy.[19] Put differently, in our model the possibility to implement type-differentiated second-period consumption taxes, which is associated with a deviation from the Friedman rule, is valuable for the planner only insofar as it is socially optimal to set $t^h < t^\ell$, i.e. to have regressive second-period consumption taxes. Looking at (56)-(57), we can see that for this to be the case, it must be that:

[19] In such a case, the sign of τ is solely determined by the complementarity/substitutability relationship between second-period consumption and labor supply. If d and L are Hicksian substitutes (complements), then the optimal τ is positive (negative).

$$
\begin{aligned}
& \frac{\lambda\alpha^{h\ell}\left(1-d^{\ell}\frac{i}{1+r}\frac{\partial\gamma^{\ell}}{\partial y^{\ell}}\right)}{\mu\pi^{\ell}\frac{\partial\widetilde{d}^{\ell}}{\partial q^{\ell}}}\left(\frac{1-d^{h\ell}\frac{i}{1+r}\frac{\partial\gamma^{h\ell}}{\partial y^{\ell}}}{1-d^{\ell}\frac{i}{1+r}\frac{\partial\gamma^{\ell}}{\partial y^{\ell}}}d^{\ell}-\frac{\gamma^{h\ell}-\gamma^{h}}{\gamma^{\ell}-\gamma^{h}}d^{h\ell}\right) \\
& -\frac{g-r}{(1+g)(1+r)}\left[\frac{\frac{\partial\gamma^{\ell}}{\partial y^{\ell}}\left(d^{\ell}\right)^{2}}{\frac{\partial\widetilde{d}^{\ell}}{\partial q^{\ell}}}+\gamma^{\ell}\right] \\
> \quad & \frac{\lambda\alpha^{h\ell}\left(1-d^{h}\frac{i}{1+r}\frac{\partial\gamma^{h}}{\partial y^{h}}\right)}{\mu\pi^{h}\frac{\partial\widetilde{d}^{h}}{\partial q^{h}}}\frac{\gamma^{h\ell}-\gamma^{\ell}}{\gamma^{\ell}-\gamma^{h}}d^{h\ell}-\frac{g-r}{(1+g)(1+r)}\left[\frac{\frac{\partial\gamma^{h}}{\partial y^{h}}\left(d^{h}\right)^{2}}{\frac{\partial\widetilde{d}^{h}}{\partial q^{h}}}+\gamma^{h}\right].
\end{aligned}
$$

Assuming $g=r$ to focus on how mimicking-deterring effects may warrant to deviate from the Friedman rule, the inequality above simplifies to:

$$
\frac{1-d^{\ell}\frac{i}{1+r}\frac{\partial\gamma^{\ell}}{\partial y^{\ell}}}{\pi^{\ell}\frac{\partial\widetilde{d}^{\ell}}{\partial q^{\ell}}}\left(\frac{1-d^{h\ell}\frac{i}{1+r}\frac{\partial\gamma^{h\ell}}{\partial y^{\ell}}}{1-d^{\ell}\frac{i}{1+r}\frac{\partial\gamma^{\ell}}{\partial y^{\ell}}}d^{\ell}-\frac{\gamma^{h\ell}-\gamma^{h}}{\gamma^{\ell}-\gamma^{h}}d^{h\ell}\right)>\frac{1-d^{h}\frac{i}{1+r}\frac{\partial\gamma^{h}}{\partial y^{h}}}{\pi^{h}\frac{\partial\widetilde{d}^{h}}{\partial q^{h}}}\frac{\gamma^{h\ell}-\gamma^{\ell}}{\gamma^{\ell}-\gamma^{h}}d^{h\ell},
$$

or, equivalently:

$$
\frac{d^{\ell}}{1-d^{\ell}\frac{i}{1+r}\frac{\partial\gamma^{\ell}}{\partial y^{\ell}}}<\frac{1}{\gamma^{\ell}-\gamma^{h}}\left[\gamma^{h\ell}-\gamma^{h}+\frac{\pi^{\ell}\frac{\partial\widetilde{d}^{\ell}}{\partial q^{\ell}}\left(1-d^{h}\frac{i}{1+r}\frac{\partial\gamma^{h}}{\partial y^{h}}\right)}{\pi^{h}\frac{\partial\widetilde{d}^{h}}{\partial q^{h}}\left(1-d^{\ell}\frac{i}{1+r}\frac{\partial\gamma^{\ell}}{\partial y^{\ell}}\right)}\left(\gamma^{h\ell}-\gamma^{\ell}\right)\right]\frac{d^{h\ell}}{1-d^{h\ell}\frac{i}{1+r}\frac{\partial\gamma^{h\ell}}{\partial y^{\ell}}}.
$$

As pointed out in Propositions 5-8, whether the inequality above holds or not hinges on circumstances such as the relative proportion of different types of agents, the complementarity/substitutability between d and L, the own-price elasticity of demand for second period consumption, the relative strength of the dependence of γ on skills and income, and the sign of the cross derivative $\partial^2\gamma/\partial y\partial w$. In particular, deviating from the Friedman rule will be more likely to be optimal when: i) the proportion of low-skilled agents is low; ii) labor supply and second-period consumption are Hicksian substitutes; iii) the own-price elasticity of demand for second period consumption is large (in absolute value); iv) the dependence of γ on income is relatively stronger than on skills; v) the cross derivative $\partial^2\gamma/\partial y\partial w$ takes a positive sign.

Regarding the effect of the terms related to the difference $g-r$, they will contribute to make the Friedman rule suboptimal when the following inequality is satisfied:

$$\frac{g-r}{(1+g)(1+r)}\left[\frac{\frac{\partial\gamma^{\ell}}{\partial y^{\ell}}\left(d^{\ell}\right)^{2}}{\frac{\partial\widetilde{d}^{\ell}}{\partial q^{\ell}}}+\gamma^{\ell}\right]<\frac{g-r}{(1+g)(1+r)}\left[\frac{\frac{\partial\gamma^{h}}{\partial y^{h}}\left(d^{h}\right)^{2}}{\frac{\partial\widetilde{d}^{h}}{\partial q^{h}}}+\gamma^{h}\right]. \tag{60}$$

From (60) we can better understand the reason why Proposition 9 could not be easily generalized to the case when $\partial\gamma/\partial y\neq 0$. In fact, if $g<r$ unambiguously strengthens the case for violating the Friedman rule when γ depends only on skills (or on skills and gross income), when $\partial\gamma/\partial y\neq 0$ this occurs only insofar as $\frac{\frac{\partial\gamma^{\ell}}{\partial y^{\ell}}\left(d^{\ell}\right)^{2}}{\frac{\partial\widetilde{d}^{\ell}}{\partial q^{\ell}}}+\gamma^{\ell}>\frac{\frac{\partial\gamma^{h}}{\partial y^{h}}\left(d^{h}\right)^{2}}{\frac{\partial\widetilde{d}^{h}}{\partial q^{h}}}+\gamma^{h}$, or equivalently $\left[\frac{\frac{\partial\gamma^{h}}{\partial y^{h}}\left(d^{h}\right)^{2}}{\frac{\partial\widetilde{d}^{h}}{\partial q^{h}}}-\frac{\frac{\partial\gamma^{\ell}}{\partial y^{\ell}}\left(d^{\ell}\right)^{2}}{\frac{\partial\widetilde{d}^{\ell}}{\partial q^{\ell}}}\right]\frac{1}{\gamma^{\ell}-\gamma^{h}}<1$. Intuitively, the efficient upward distortion on t^{ℓ} must be larger than that on t^{h} in order to make it desirable to set $i>0$.

Finally, suppose that the value taken by (57) exceeds the value taken by (56), so that it is optimal to set $i>0$ and violate the Friedman rule. One can then ask what this implies for the sign of the optimal τ. Even though τ cannot be unambiguously signed, its sign is more likely to be positive when the difference $\gamma^{\ell}-\gamma^{h}$ is large and the the difference $t^{\ell}-t^{h}$ is small. On the other hand, a negative τ is more likely to be optimal when the difference $\gamma^{\ell}-\gamma^{h}$ is small and the the difference $t^{\ell}-t^{h}$ is large.

7 Summary and conclusion

This paper has modeled an overlapping-generations economy à la Samuelson (1958) with money wherein money holdings are rationalized by a version of the Clower cash-in-advance constraint. It has allowed for two correlated dimensions of heterogeneity. Some agents are more skilled and more financially connected than others. This means that they have a higher earning ability and require a smaller cash reserve to mediate their expenditures. The government has information on individuals' incomes and anonymous expenditures; allowing it to levy nonlinear income and linear commodity taxes. Money supply increases, or contracts, at a fixed rate per year through lump-sum money transfers to individuals. Within this framework, the paper has studied the nature of the economy's temporal equilibrium as well as its steady state. It has also characterized the informationally constrained Pareto-efficient allocations of this economy.

An important message of the paper is that notwithstanding the fiscal authority's ability to levy nonlinear income taxes, it is unable to fully replicate or neutralize the redistributive implications of monetary policy. More specifically, *for a given monetary rate of growth,* the fiscal authority can offset the redistributive effects of who gets the extra money (or loses the money that is withdrawn from the economy). It can adjust the individuals' income tax payments and ensure that all agents will continue to enjoy the same level of welfare. The problem lies with the redistributive implications of monetary growth rate. This the fiscal authority cannot fully neutralize. The reason is that, unlike a change in the tax rate, a change in monetary growth rate changes the intertemporal price of consumption goods differently for different individual types. It is this property that differentiates monetary policy from fiscal policy in terms of their redistributive potential. In turn, this property arises because of the heterogeneity in financial connectedness of the agents.

Another contribution of the paper is to show that the Friedman rule may not be part of an optimal policy. Deviating from the Friedman rule enables the policy-maker to implement type-differentiated second-period consumption taxes. More precisely, and due to the non-negativity constraint on the nominal interest rate, it allows the government to implement regressive second-period consumption taxes, which we have shown can be socially optimal even though the social welfare function favors redistribution from the high- to the low-ability agents.

Having characterized the optimal nominal interest rate and the optimal tax on second-period consumption, the paper has shown that each has a unique role in determining the optimal "effective tax" on second-period consumption. Moreover, the paper has highlighted that whether or not it is optimal to abide by the Friedman rule depends on circumstances such as the relative proportion of different types of agents in the population, the complementarity/substitutability between labor supply and second-period consumption, the own-price elasticity of demand for second period consumption, the relative strength of the dependence of γ on skills and income, the complementarity/substitutability between skills and income in the γ-function, and the sign of the difference between the rate of population growth and the real rate of interest. In particular, deviating from the Friedman rule is more likely to be part of an optimal policy when: i) the proportion of low-skilled agents is low; ii) labor supply and second-period consumption are Hicksian substitutes; iii) the own-price elasticity of demand for second period consumption is large (in absolute value); iv) the dependence of γ on income is relatively stronger than on skills; v) the cross derivative $\partial^2\gamma/\partial y\partial w$ takes a positive sign. Regarding any difference which may occur between the rate of population growth g and the real rate of interest r, the paper has shown that its impact on the desirability to abide by the Friedman rule is in general ambiguous. However, at least when γ depends only on skills or when it depends on skills and pre-tax labor income, a negative sign of the difference $g - r$ calls for deviating from the Friedman rule.

Two other results concern the special cases wherein skills are the sole source of heterogeneity and when the degree of financial connectedness depends solely on skills but not on incomes.

In the first case, the fiscal authority is able to neutralize the effects of a change in the rate of monetary growth. Under this circumstance, the optimal monetary growth rate is not unique. A continuum of values for the monetary growth rate and the tax on the second-period consumption, coupled with supporting income tax rates, maximizes social welfare. Abiding by the Friedman rule represents in this case only one among a continuum of equivalent optima.

In the second case, abiding by the Friedman rule is necessarily part of an optimal policy, at least provided that $g > r$.

In conclusion, we should emphasize that the paper has completely ignored the macroeconomic issues associated with monetary and fiscal policies. Questions such as stabilization, unemployment, sticky prices, and the like have not been touched in this study not because they are unimportant but simply because they are outside the purview of the current study.[20]

[20] Some of these issues are discussed by Correia *et al.* (2008) in a dynamic Ramsey setting. They show that sticky prices are irrelevant for the conduct of monetary policy if fiscal instruments are not restricted.

Appendix A

Proof of equations (49)-(50): Given the redundancy of one of the redistributive instruments b^h and b^ℓ, it is sufficient to carry out our optimization with respect to only b^h or b^ℓ. Without any loss of generality, we will choose b^h. The first-order conditions associated with the Lagrangian (48) are then given by (allowing for the possibility that γ depends on skills and on either pre-tax income I or aggregate disposable income y):

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial I^h} &= \left(\delta^h + \lambda\right)\left(\frac{\partial v^h}{\partial I^h} + \frac{\partial v^h}{\partial q^h}\frac{\partial q^h}{\partial \gamma^h}\frac{\partial \gamma^h}{\partial I^h}\right) \\
&\quad -\eta\frac{1}{1+r}\left(\frac{g-r}{1+g}+i\right)\pi^h\left[\gamma^h\frac{\partial d^h}{\partial I^h} + \frac{\partial \gamma^h}{\partial I^h}\left(d^h + \gamma^h\frac{\partial d^h}{\partial q^h}\frac{\partial q^h}{\partial \gamma^h}\right)\right] \\
&\quad +\mu\pi^h\left[1 + \frac{\tau}{1+r}\left(\frac{\partial d^h}{\partial I^h} + \frac{\partial d^h}{\partial q^h}\frac{\partial q^h}{\partial \gamma^h}\frac{\partial \gamma^h}{\partial I^h}\right)\right] \\
&= 0 \qquad \text{(A1)} \\
\frac{\partial \mathcal{L}}{\partial I^\ell} &= \delta^\ell\left(\frac{\partial v^\ell}{\partial I^\ell} + \frac{\partial v^\ell}{\partial q^\ell}\frac{\partial q^\ell}{\partial \gamma^\ell}\frac{\partial \gamma^\ell}{\partial I^\ell}\right) - \lambda\left(\frac{\partial v^{h\ell}}{\partial I^\ell} + \frac{\partial v^{h\ell}}{\partial q^{h\ell}}\frac{\partial q^{h\ell}}{\partial \gamma^{h\ell}}\frac{\partial \gamma^{h\ell}}{\partial I^\ell}\right) \\
&\quad -\eta\frac{1}{1+r}\left(\frac{g-r}{1+g}+i\right)\pi^\ell\left[\gamma^\ell\frac{\partial d^\ell}{\partial I^\ell} + \frac{\partial \gamma^\ell}{\partial I^\ell}\left(d^\ell + \gamma^\ell\frac{\partial d^\ell}{\partial q^\ell}\frac{\partial q^\ell}{\partial \gamma^\ell}\right)\right] \\
&\quad +\mu\pi^\ell\left[1 + \frac{\tau}{1+r}\left(\frac{\partial d^\ell}{\partial I^\ell} + \frac{\partial d^\ell}{\partial q^\ell}\frac{\partial q^\ell}{\partial \gamma^\ell}\frac{\partial \gamma^\ell}{\partial I^\ell}\right)\right] \\
&= 0 \qquad \text{(A2)} \\
\frac{\partial \mathcal{L}}{\partial z^h} &= \left(\delta^h + \lambda\right)\left(\frac{\partial v^h}{\partial y^h} + \frac{\partial v^h}{\partial q^h}\frac{\partial q^h}{\partial \gamma^h}\frac{\partial \gamma^h}{\partial y^h}\right) \\
&\quad -\eta\frac{1}{1+r}\left(\frac{g-r}{1+g}+i\right)\pi^h\left[\gamma^h\frac{\partial d^h}{\partial y^h} + \frac{\partial \gamma^h}{\partial y^h}\left(d^h + \gamma^h\frac{\partial d^h}{\partial q^h}\frac{\partial q^h}{\partial \gamma^h}\right)\right] \\
&\quad +\mu\pi^h\left[-1 + \frac{\tau}{1+r}\left(\frac{\partial d^h}{\partial y^h} + \frac{\partial d^h}{\partial q^h}\frac{\partial q^h}{\partial \gamma^h}\frac{\partial \gamma^h}{\partial y^h}\right)\right] \\
&= 0 \qquad \text{(A3)}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial z^\ell} &= \delta^\ell \left(\frac{\partial v^\ell}{\partial y^\ell} + \frac{\partial v^\ell}{\partial q^\ell}\frac{\partial q^\ell}{\partial \gamma^\ell}\frac{\partial \gamma^\ell}{\partial y^\ell}\right) - \lambda \left(\frac{\partial v^{h\ell}}{\partial y^\ell} + \frac{\partial v^{h\ell}}{\partial q^{h\ell}}\frac{\partial q^{h\ell}}{\partial \gamma^{h\ell}}\frac{\partial \gamma^{h\ell}}{\partial y^\ell}\right) \\
&\quad -\eta \frac{1}{1+r}\left(\frac{g-r}{1+g}+i\right)\pi^\ell \left[\gamma^\ell \frac{\partial d^\ell}{\partial y^\ell} + \frac{\partial \gamma^\ell}{\partial y^\ell}\left(d^\ell + \gamma^\ell \frac{\partial d^\ell}{\partial q^\ell}\frac{\partial q^\ell}{\partial \gamma^\ell}\right)\right] \\
&\quad +\mu \pi^\ell \left[-1 + \frac{\tau}{1+r}\left(\frac{\partial d^\ell}{\partial y^\ell} + \frac{\partial d^\ell}{\partial q^\ell}\frac{\partial q^\ell}{\partial \gamma^\ell}\frac{\partial \gamma^\ell}{\partial y^\ell}\right)\right] \\
&= 0 \qquad \text{(A4)} \\
\frac{\partial \mathcal{L}}{\partial b^h} &= \left(\delta^h + \lambda\right)\left(\frac{\partial v^h}{\partial y^h} + \frac{\partial v^h}{\partial q^h}\frac{\partial q^h}{\partial \gamma^h}\frac{\partial \gamma^h}{\partial y^h}\right) \\
&\quad +\eta \pi^h \left\{1 - \frac{1}{1+r}\left(\frac{g-r}{1+g}+i\right)\left[\gamma^h \frac{\partial d^h}{\partial y^h} + \frac{\partial \gamma^h}{\partial y^h}\left(d^h + \gamma^h \frac{\partial d^h}{\partial q^h}\frac{\partial q^h}{\partial \gamma^h}\right)\right]\right\} \\
&\quad +\mu \pi^h \frac{\tau}{1+r}\left(\frac{\partial d^h}{\partial y^h} + \frac{\partial d^h}{\partial q^h}\frac{\partial q^h}{\partial \gamma^h}\frac{\partial \gamma^h}{\partial y^h}\right) \\
&= 0 \qquad \text{(A5)} \\
\frac{\partial \mathcal{L}}{\partial \tau} &= \sum_{j=\ell,h} \delta^j \frac{\partial v^j}{\partial \tau} + \lambda \left(\frac{\partial v^h}{\partial \tau} - \frac{\partial v^{h\ell}}{\partial \tau}\right) \\
&\quad -\eta \frac{1}{1+r}\left(\frac{g-r}{1+g}+i\right)\sum_{j=\ell,h}\pi^j \gamma^j \frac{\partial d^j}{\partial \tau} + \frac{\mu}{1+r}\sum_{j=\ell,h}\pi^j \left(d^j + \tau \frac{\partial d^j}{\partial \tau}\right) \\
&= 0, \qquad \text{(A6)} \\
\frac{\partial \mathcal{L}}{\partial i} &= \sum_{j=\ell,h} \delta^j \frac{\partial v^j}{\partial i} + \lambda \left(\frac{\partial v^h}{\partial i} - \frac{\partial v^{h\ell}}{\partial i}\right) \\
&\quad -\eta \frac{1}{1+r}\left[\left(\frac{g-r}{1+g}+i\right)\sum_{j=\ell,h}\pi^j \gamma^j \frac{\partial d^j}{\partial i} + \sum_{j=\ell,h}\pi^j \gamma^j d^j\right] + \frac{\mu \tau}{1+r}\sum_{j=\ell,h}\pi^j \frac{\partial d^j}{\partial i} \\
&= 0, \qquad \text{(A7)}
\end{aligned}$$

where comparing equation (A3) with (A5) reveals that $\mu = -\eta$.

Now substitute for i from (27) in (37) to get

$$q^{jk} = \frac{1+\tau+\gamma^{jk} i}{1+r}. \qquad \text{(A8)}$$

Differentiate equations (42) and (A8) with respect to τ and i to get

$$\frac{\partial q^j}{\partial \tau} = \frac{\partial q^{jk}}{\partial \tau} = \frac{1}{1+r}, \tag{A9}$$

$$\frac{\partial q^j}{\partial i} = \frac{\gamma^j}{1+r}, \tag{A10}$$

$$\frac{\partial q^{jk}}{\partial i} = \frac{\gamma^{jk}}{1+r}. \tag{A11}$$

Next differentiate v^j and v^{jk}, as specified by equations (34) and (40), with respect to z^j, z^k, τ and i. We get,

$$\frac{\partial v^j}{\partial z^j}|_{\tau,i,b^j,I^j,\gamma^j} = \frac{\partial v^j}{\partial b^j}|_{\tau,i,z^j,I^j,\gamma^j} = \frac{\partial v^j}{\partial y^j}|_{q^j,I^j} \equiv \alpha^j, \tag{A12}$$

$$\frac{\partial v^{jk}}{\partial z^k}|_{\tau,i,b^k,I^k,\gamma^{jk}} = \frac{\partial v^{jk}}{\partial b^k}|_{\tau,i,z^k,I^k,\gamma^{jk}} = \frac{\partial v^{jk}}{\partial y^k}|_{q^{jk},I^k} \equiv \alpha^{jk}. \tag{A13}$$

Roy's identity then implies,

$$\frac{\partial v^j}{\partial \tau}|_{\theta,b^j,z^j,I^j} = \frac{\partial v^j}{\partial q^j}|_{y^j,I^j}\frac{\partial q^j}{\partial \tau}|_i = \frac{-\alpha^j d^j}{1+r}, \tag{A14}$$

$$\frac{\partial v^{jk}}{\partial \tau}|_{\theta,b^k,z^k,I^k} = \frac{\partial v^{jk}}{\partial q^{jk}}|_{y^k,I^k}\frac{\partial q^{jk}}{\partial \tau}|_i = \frac{-\alpha^{jk} d^{jk}}{1+r}, \tag{A15}$$

$$\frac{\partial v^j}{\partial i}|_{\tau,b^j,z^j,I^j} = \frac{\partial v^j}{\partial q^j}|_{y^j,I^j}\frac{\partial q^j}{\partial i}|_\tau = \frac{-\gamma^j\alpha^j d^j}{1+r}, \tag{A16}$$

$$\frac{\partial v^{jk}}{\partial i}|_{\tau,b^k,z^k,I^k} = \frac{\partial v^{jk}}{\partial q^{jk}}|_{y^k,I^k}\frac{\partial q^{jk}}{\partial i}|_\tau = \frac{-\gamma^{jk}\alpha^{jk} d^{jk}}{1+r}. \tag{A17}$$

Use eqs. (A12)–(A17) to simplify the first-order conditions (A1)–(A7) as

$$\begin{aligned}
&\left(\delta^h+\lambda\right)\left(\frac{\partial v^h}{\partial I^h} - \alpha^h d^h \frac{i}{1+r}\frac{\partial \gamma^h}{\partial I^h}\right) + \mu\pi^h\left\{1+\left[\frac{\gamma^h}{1+r}\left(\frac{g-r}{1+g}+i\right)+\frac{\tau}{1+r}\right]\frac{\partial d^h}{\partial I^h}\right\}\\
&+\mu\pi^h\frac{\partial \gamma^h}{\partial I^h}\left[\frac{\tau}{1+r}\frac{i}{1+r}\frac{\partial d^h}{\partial q^h} + \frac{1}{1+r}\left(\frac{g-r}{1+g}+i\right)\left(d^h+\frac{\gamma^h i}{1+r}\right)\frac{\partial d^h}{\partial q^h}\right]\\
=\ & 0
\end{aligned} \tag{A18}$$

$$
\begin{aligned}
& \delta^{\ell}\left(\frac{\partial v^{\ell}}{\partial I^{\ell}}-\alpha^{\ell} d^{\ell} \frac{i}{1+r} \frac{\partial \gamma^{\ell}}{\partial I^{\ell}}\right)-\lambda\left(\frac{\partial v^{h\ell}}{\partial I^{\ell}}-\alpha^{h\ell} d^{h\ell} \frac{i}{1+r} \frac{\partial \gamma^{h\ell}}{\partial I^{h\ell}}\right) \\
& +\mu \pi^{\ell}\left\{1+\left[\gamma^{\ell} \frac{1}{1+r}\left(\frac{g-r}{1+g}+i\right)+\frac{\tau}{1+r}\right] \frac{\partial d^{\ell}}{\partial I^{\ell}}\right\} \\
& +\mu \pi^{h} \frac{\partial \gamma^{\ell}}{\partial I^{\ell}}\left[\frac{\tau}{1+r} \frac{i}{1+r} \frac{\partial d^{\ell}}{\partial q^{\ell}}+\frac{1}{1+r}\left(\frac{g-r}{1+g}+i\right)\left(d^{\ell}+\frac{\gamma^{\ell} i}{1+r}\right) \frac{\partial d^{\ell}}{\partial q^{\ell}}\right] \\
= \; & 0 \qquad \text{(A19)}
\end{aligned}
$$

$$
\begin{aligned}
& \left(\delta^{h}+\lambda\right) \alpha^{h}\left(1-d^{h} \frac{i}{1+r} \frac{\partial \gamma^{h}}{\partial y^{h}}\right) \\
& +\mu \frac{1}{1+r}\left(\frac{g-r}{1+g}+i\right) \pi^{h}\left[\gamma^{h} \frac{\partial d^{h}}{\partial y^{h}}+\frac{\partial \gamma^{h}}{\partial y^{h}}\left(d^{h}+\gamma^{h} \frac{\partial d^{h}}{\partial q^{h}} \frac{\partial q^{h}}{\partial \gamma^{h}}\right)\right] \\
& +\mu \pi^{h}\left[-1+\frac{\tau}{1+r}\left(\frac{\partial d^{h}}{\partial y^{h}}+\frac{\partial d^{h}}{\partial q^{h}} \frac{\partial q^{h}}{\partial \gamma^{h}} \frac{\partial \gamma^{h}}{\partial y^{h}}\right)\right] \\
= \; & 0 \qquad \text{(A20)}
\end{aligned}
$$

$$
\begin{aligned}
& \delta^{\ell} \alpha^{\ell}\left(1-d^{\ell} \frac{i}{1+r} \frac{\partial \gamma^{\ell}}{\partial y^{\ell}}\right)-\lambda \alpha^{h\ell}\left(1-d^{h\ell} \frac{i}{1+r} \frac{\partial \gamma^{h\ell}}{\partial y^{\ell}}\right) \\
& +\mu \frac{1}{1+r}\left(\frac{g-r}{1+g}+i\right) \pi^{\ell}\left[\gamma^{\ell} \frac{\partial d^{\ell}}{\partial y^{\ell}}+\frac{\partial \gamma^{\ell}}{\partial y^{\ell}}\left(d^{\ell}+\gamma^{\ell} \frac{\partial d^{\ell}}{\partial q^{\ell}} \frac{\partial q^{\ell}}{\partial \gamma^{\ell}}\right)\right] \\
& +\mu \pi^{\ell}\left[-1+\frac{\tau}{1+r}\left(\frac{\partial d^{\ell}}{\partial y^{\ell}}+\frac{\partial d^{\ell}}{\partial q^{\ell}} \frac{\partial q^{\ell}}{\partial \gamma^{\ell}} \frac{\partial \gamma^{\ell}}{\partial y^{\ell}}\right)\right] \\
= \; & 0 \qquad \text{(A21)}
\end{aligned}
$$

$$
\begin{aligned}
& \lambda \alpha^{h\ell} d^{h\ell}-\delta^{\ell} \alpha^{\ell} d^{\ell}-\left(\delta^{h}+\lambda\right) \alpha^{h} d^{h} \\
& +\mu\left[\sum_{j=\ell, h} \pi^{j} d^{j}+\frac{1}{1+r}\left(\frac{g-r}{1+g}+i\right) \sum_{j=\ell, h} \pi^{j} \gamma^{j} \frac{\partial d^{j}}{\partial q^{j}}+\frac{\tau}{1+r} \sum_{j=\ell, h} \pi^{j} \frac{\partial d^{j}}{\partial q^{j}}\right] \\
= \; & 0 \qquad \text{(A22)}
\end{aligned}
$$

$$
\begin{aligned}
& \lambda \alpha^{h\ell} \gamma^{h\ell} d^{h\ell}-\delta^{\ell} \alpha^{\ell} \gamma^{\ell} d^{\ell}-\left(\delta^{h}+\lambda\right) \alpha^{h} \gamma^{h} d^{h} \\
& +\mu\left[\sum_{j=\ell, h} \pi^{j} \gamma^{j} d^{j}+\frac{1}{1+r}\left(\frac{g-r}{1+g}+i\right) \sum_{j=\ell, h} \pi^{j}\left(\gamma^{j}\right)^{2} \frac{\partial d^{j}}{\partial q^{j}}+\frac{\tau}{1+r} \sum_{j=\ell, h} \pi^{j} \gamma^{j} \frac{\partial d^{j}}{\partial q^{j}}\right] \\
= \; & 0. \qquad \text{(A23)}
\end{aligned}
$$

Multiply eq. (A20) by $d^h / \left(1 - d^h \frac{i}{1+r} \frac{\partial \gamma^h}{\partial y^h}\right)$ and eq. (A21) by $d^\ell / \left(1 - d^\ell \frac{i}{1+r} \frac{\partial \gamma^\ell}{\partial y^\ell}\right)$, then add the resulting two equations to (A22). Letting $\widetilde{d^j}$ denote the compensated version of d^j and using the Slutsky equation gives:

$$\begin{aligned}
&\lambda\alpha^{h\ell}\left(d^{h\ell}-\frac{1-d^{h\ell}\frac{i}{1+r}\frac{\partial\gamma^{h\ell}}{\partial y^\ell}}{1-d^\ell\frac{i}{1+r}\frac{\partial\gamma^\ell}{\partial y^\ell}}d^\ell\right)\\
&+\mu\left[\frac{1}{1+r}\left(\frac{g-r}{1+g}+i\right)\sum_{j=\ell,h}\pi^j d^j\frac{\partial\gamma^j}{\partial y^j}\frac{d^j}{1-d^j\frac{i}{1+r}\frac{\partial\gamma^j}{\partial y^j}}-\sum_{j=\ell,h}\pi^j d^j\frac{d^j\frac{i}{1+r}\frac{\partial\gamma^j}{\partial y^j}}{1-d^j\frac{i}{1+r}\frac{\partial\gamma^j}{\partial y^j}}\right]\\
&+\frac{\mu}{1+r}\left[\frac{1}{1+r}\left(\frac{g-r}{1+g}+i\right)\sum_{j=\ell,h}\pi^j\gamma^j\frac{\partial\widetilde{d^j}}{\partial q^j}\frac{1}{1-d^j\frac{i}{1+r}\frac{\partial\gamma^j}{\partial y^j}}+\tau\sum_{j=\ell,h}\pi^j\frac{\partial\widetilde{d^j}}{\partial q^j}\frac{1}{1-d^j\frac{i}{1+r}\frac{\partial\gamma^j}{\partial y^j}}\right]\\
=\;&0,
\end{aligned}$$

or,

$$\begin{aligned}
&\lambda\alpha^{h\ell}\left(d^{h\ell}-\frac{1-d^{h\ell}\frac{i}{1+r}\frac{\partial\gamma^{h\ell}}{\partial y^\ell}}{1-d^\ell\frac{i}{1+r}\frac{\partial\gamma^\ell}{\partial y^\ell}}d^\ell\right)+\mu\left[\sum_{j=\ell,h}\frac{\pi^j\left(d^j\right)^2\frac{\partial\gamma^j}{\partial y^j}\frac{1}{1+r}}{1-d^j\frac{i}{1+r}\frac{\partial\gamma^j}{\partial y^j}}\left(\frac{g-r}{1+g}+i-i\right)\right]\\
&+\frac{\mu}{1+r}\left[\left(\frac{g-r}{1+g}+i\right)\sum_{j=\ell,h}\pi^j\gamma^j\frac{\partial\widetilde{d^j}}{\partial q^j}\frac{1}{1-d^j\frac{i}{1+r}\frac{\partial\gamma^j}{\partial y^j}}+\tau\sum_{j=\ell,h}\pi^j\frac{\partial\widetilde{d^j}}{\partial q^j}\frac{1}{1-d^j\frac{i}{1+r}\frac{\partial\gamma^j}{\partial y^j}}\right]\\
=\;&0.
\end{aligned}$$

Simplifying terms and multiplying by $(1+r)/\mu$ gives:

$$\begin{aligned}
&\frac{(1+r)\lambda\alpha^{h\ell}}{\mu}\left(d^{h\ell}-\frac{1-d^{h\ell}\frac{i}{1+r}\frac{\partial\gamma^{h\ell}}{\partial y^\ell}}{1-d^\ell\frac{i}{1+r}\frac{\partial\gamma^\ell}{\partial y^\ell}}d^\ell\right)+\frac{g-r}{1+g}\sum_{j=\ell,h}\frac{\pi^j\left(d^j\right)^2\frac{\partial\gamma^j}{\partial y^j}}{1-d^j\frac{i}{1+r}\frac{\partial\gamma^j}{\partial y^j}}\\
&+\left(\frac{g-r}{1+g}+i\right)\sum_{j=\ell,h}\pi^j\gamma^j\frac{\partial\widetilde{d^j}}{\partial q^j}\frac{1}{1-d^j\frac{i}{1+r}\frac{\partial\gamma^j}{\partial y^j}}+\tau\sum_{j=\ell,h}\pi^j\frac{\partial\widetilde{d^j}}{\partial q^j}\frac{1}{1-d^j\frac{i}{1+r}\frac{\partial\gamma^j}{\partial y^j}}\\
=\;&0. \qquad \text{(A24)}
\end{aligned}$$

Then multiply eq. (A20) by $\gamma^h d^h / \left(1 - d^h \frac{i}{1+r} \frac{\partial \gamma^h}{\partial y^h}\right)$ and eq. (A21) by $\gamma^\ell d^\ell / \left(1 - d^\ell \frac{i}{1+r} \frac{\partial \gamma^\ell}{\partial y^\ell}\right)$, and add the resulting two equations to (A23) to get

$$\begin{aligned}
&\mu \frac{1}{1+r}\left(\frac{g-r}{1+g}+i\right)\sum_{j=\ell,h}\pi^j \frac{\partial \gamma^j}{\partial y^j} d^j \frac{\gamma^j d^j}{1-d^j \frac{i}{1+r}\frac{\partial \gamma^j}{\partial y^j}} - \mu \sum_{j=\ell,h}\pi^j \gamma^j d^j \frac{d^j \frac{i}{1+r}\frac{\partial \gamma^j}{\partial y^j}}{1-d^j \frac{i}{1+r}\frac{\partial \gamma^j}{\partial y^j}} \\
&+ \lambda \alpha^{h\ell}\left(\gamma^{h\ell} d^{h\ell} - \frac{1-d^{h\ell}\frac{i}{1+r}\frac{\partial \gamma^{h\ell}}{\partial y^\ell}}{1-d^\ell \frac{i}{1+r}\frac{\partial \gamma^\ell}{\partial y^\ell}}\gamma^\ell d^\ell\right) \\
&+ \frac{\mu}{1+r}\left[\left(\frac{g-r}{1+g}+i\right)\sum_{j=\ell,h}\pi^j \left(\gamma^j\right)^2 \frac{\partial \widetilde{d^j}}{\partial q^j}\frac{1}{1-d^j \frac{i}{1+r}\frac{\partial \gamma^j}{\partial y^j}} + \tau \sum_{j=\ell,h}\pi^j \gamma^j \frac{\partial \widetilde{d^j}}{\partial q^j}\frac{1}{1-d^j \frac{i}{1+r}\frac{\partial \gamma^j}{\partial y^j}}\right] \\
&= 0.
\end{aligned}$$

Simplifying terms and multiplying by $(1+r)/\mu$ gives:

$$\begin{aligned}
&\frac{g-r}{1+g}\sum_{j=\ell,h}\frac{\pi^j \frac{\partial \gamma^j}{\partial y^j}\left(d^j\right)^2 \gamma^j}{1-d^j \frac{i}{1+r}\frac{\partial \gamma^j}{\partial y^j}} + \frac{(1+r)\lambda}{\mu}\alpha^{h\ell}\left(\gamma^{h\ell} d^{h\ell} - \frac{1-d^{h\ell}\frac{i}{1+r}\frac{\partial \gamma^{h\ell}}{\partial y^\ell}}{1-d^\ell \frac{i}{1+r}\frac{\partial \gamma^\ell}{\partial y^\ell}}\gamma^\ell d^\ell\right) \\
&+\left(\frac{g-r}{1+g}+i\right)\sum_{j=\ell,h}\pi^j \left(\gamma^j\right)^2 \frac{\partial \widetilde{d^j}}{\partial q^j}\frac{1}{1-d^j \frac{i}{1+r}\frac{\partial \gamma^j}{\partial y^j}} + \tau \sum_{j=\ell,h}\pi^j \gamma^j \frac{\partial \widetilde{d^j}}{\partial q^j}\frac{1}{1-d^j \frac{i}{1+r}\frac{\partial \gamma^j}{\partial y^j}} \\
&= 0. \qquad \text{(A25)}
\end{aligned}$$

Next write equations (A24) and (A25) in matrix form as

$$\begin{bmatrix} \sum_{j=\ell,h}\pi^j \frac{\partial \widetilde{d^j}}{\partial q^j}\frac{1}{1-d^j \frac{i}{1+r}\frac{\partial \gamma^j}{\partial y^j}} & \sum_{j=\ell,h}\pi^j \gamma^j \frac{\partial \widetilde{d^j}}{\partial q^j}\frac{1}{1-d^j \frac{i}{1+r}\frac{\partial \gamma^j}{\partial y^j}} \\ \sum_{j=\ell,h}\pi^j \gamma^j \frac{\partial \widetilde{d^j}}{\partial q^j}\frac{1}{1-d^j \frac{i}{1+r}\frac{\partial \gamma^j}{\partial y^j}} & \sum_{j=\ell,h}\pi^j \left(\gamma^j\right)^2 \frac{\partial \widetilde{d^j}}{\partial q^j}\frac{1}{1-d^j \frac{i}{1+r}\frac{\partial \gamma^j}{\partial y^j}} \end{bmatrix}\begin{bmatrix} \tau \\ \frac{g-r}{1+g}+i \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{(1+r)\lambda}{\mu}\alpha^{h\ell}\left(d^{h\ell} - \frac{1-d^{h\ell}\frac{i}{1+r}\frac{\partial \gamma^{h\ell}}{\partial y^\ell}}{1-d^\ell \frac{i}{1+r}\frac{\partial \gamma^\ell}{\partial y^\ell}} d^\ell\right) - \frac{g-r}{1+g}\sum_{j=\ell,h}\frac{\pi^j \left(d^j\right)^2 \frac{\partial \gamma^j}{\partial y^j}}{1-d^j \frac{i}{1+r}\frac{\partial \gamma^j}{\partial y^j}} \\ -\frac{(1+r)\lambda}{\mu}\alpha^{h\ell}\left(\gamma^{h\ell} d^{h\ell} - \frac{1-d^{h\ell}\frac{i}{1+r}\frac{\partial \gamma^{h\ell}}{\partial y^\ell}}{1-d^\ell \frac{i}{1+r}\frac{\partial \gamma^\ell}{\partial y^\ell}}\gamma^\ell d^\ell\right) - \frac{g-r}{1+g}\sum_{j=\ell,h}\frac{\pi^j \left(d^j\right)^2 \frac{\partial \gamma^j}{\partial y^j}\gamma^j}{1-d^j \frac{i}{1+r}\frac{\partial \gamma^j}{\partial y^j}} \end{bmatrix}. \qquad \text{(A26)}$$

The determinant of the 2×2 matrix on the left-hand side of (A26) is

$$\begin{aligned}
& \left(\sum_{j=\ell,h} \pi^j \frac{\partial \widetilde{d}^j}{\partial q^j} \frac{1}{1-d^j \frac{i}{1+r} \frac{\partial \gamma^j}{\partial y^j}}\right) \left(\sum_{j=\ell,h} \pi^j \left(\gamma^j\right)^2 \frac{\partial \widetilde{d}^j}{\partial q^j} \frac{1}{1-d^j \frac{i}{1+r} \frac{\partial \gamma^j}{\partial y^j}}\right) \\
& - \left(\sum_{j=\ell,h} \pi^j \gamma^j \frac{\partial \widetilde{d}^j}{\partial q^j} \frac{1}{1-d^j \frac{i}{1+r} \frac{\partial \gamma^j}{\partial y^j}}\right)^2 \\
= \; & \pi^\ell \pi^h \frac{\partial \widetilde{d}^\ell}{\partial q^\ell} \frac{\partial \widetilde{d}^h}{\partial q^h} \frac{1}{1-d^\ell \frac{i}{1+r} \frac{\partial \gamma^\ell}{\partial y^\ell}} \frac{1}{1-d^h \frac{i}{1+r} \frac{\partial \gamma^h}{\partial y^h}} \left(\gamma^\ell - \gamma^h\right)^2 > 0.
\end{aligned}$$

Denoting this determinant by Δ and using Cramer's rule, we have:

$$\begin{aligned}
\Delta \tau = \; & \left(\sum_{j=\ell,h} \pi^j \left(\gamma^j\right)^2 \frac{\partial \widetilde{d}^j}{\partial q^j} \frac{1}{1-d^j \frac{i}{1+r} \frac{\partial \gamma^j}{\partial y^j}}\right) \left[-\frac{(1+r)\lambda}{\mu} \alpha^{h\ell} \left(d^{h\ell} - \frac{1-d^{h\ell} \frac{i}{1+r} \frac{\partial \gamma^{h\ell}}{\partial y^\ell}}{1-d^\ell \frac{i}{1+r} \frac{\partial \gamma^\ell}{\partial y^\ell}} d^\ell\right)\right] \\
& - \left(\sum_{j=\ell,h} \pi^j \left(\gamma^j\right)^2 \frac{\partial \widetilde{d}^j}{\partial q^j} \frac{1}{1-d^j \frac{i}{1+r} \frac{\partial \gamma^j}{\partial y^j}}\right) \frac{g-r}{1+g} \sum_{j=\ell,h} \frac{\pi^j \left(d^j\right)^2 \frac{\partial \gamma^j}{\partial y^j}}{1-d^j \frac{i}{1+r} \frac{\partial \gamma^j}{\partial y^j}} \\
& - \left(\sum_{j=\ell,h} \pi^j \gamma^j \frac{\partial \widetilde{d}^j}{\partial q^j} \frac{1}{1-d^j \frac{i}{1+r} \frac{\partial \gamma^j}{\partial y^j}}\right) \left[-\frac{(1+r)\lambda}{\mu} \alpha^{h\ell} \left(\gamma^{h\ell} d^{h\ell} - \frac{1-d^{h\ell} \frac{i}{1+r} \frac{\partial \gamma^{h\ell}}{\partial y^\ell}}{1-d^\ell \frac{i}{1+r} \frac{\partial \gamma^\ell}{\partial y^\ell}} \gamma^\ell d^\ell\right)\right] \\
& + \left(\sum_{j=\ell,h} \pi^j \gamma^j \frac{\partial \widetilde{d}^j}{\partial q^j} \frac{1}{1-d^j \frac{i}{1+r} \frac{\partial \gamma^j}{\partial y^j}}\right) \frac{g-r}{1+g} \sum_{j=\ell,h} \frac{\pi^j \left(d^j\right)^2 \frac{\partial \gamma^j}{\partial y^j} \gamma^j}{1-d^j \frac{i}{1+r} \frac{\partial \gamma^j}{\partial y^j}} \\
= \; & \frac{(1+r)\lambda \alpha^{h\ell}}{\mu} \left[\left(\gamma^{h\ell} d^{h\ell} - \frac{1-d^{h\ell} \frac{i}{1+r} \frac{\partial \gamma^{h\ell}}{\partial y^\ell}}{1-d^\ell \frac{i}{1+r} \frac{\partial \gamma^\ell}{\partial y^\ell}} \gamma^\ell d^\ell\right) \left(\sum_{j=\ell,h} \pi^j \gamma^j \frac{\partial \widetilde{d}^j}{\partial q^j} \frac{1}{1-d^j \frac{i}{1+r} \frac{\partial \gamma^j}{\partial y^j}}\right)\right] \\
& - \frac{(1+r)\lambda \alpha^{h\ell}}{\mu} \left(d^{h\ell} - \frac{1-d^{h\ell} \frac{i}{1+r} \frac{\partial \gamma^{h\ell}}{\partial y^\ell}}{1-d^\ell \frac{i}{1+r} \frac{\partial \gamma^\ell}{\partial y^\ell}} d^\ell\right) \left(\sum_{j=\ell,h} \pi^j \left(\gamma^j\right)^2 \frac{\partial \widetilde{d}^j}{\partial q^j} \frac{1}{1-d^j \frac{i}{1+r} \frac{\partial \gamma^j}{\partial y^j}}\right) \\
& + \frac{g-r}{1+g} \left[\left(\sum_{j=\ell,h} \frac{\pi^j \left(d^j\right)^2 \frac{\partial \gamma^j}{\partial y^j} \gamma^j}{1-d^j \frac{i}{1+r} \frac{\partial \gamma^j}{\partial y^j}}\right) \left(\sum_{j=\ell,h} \pi^j \gamma^j \frac{\partial \widetilde{d}^j}{\partial q^j} \frac{1}{1-d^j \frac{i}{1+r} \frac{\partial \gamma^j}{\partial y^j}}\right)\right] \\
& - \frac{g-r}{1+g} \left(\sum_{j=\ell,h} \frac{\pi^j \left(d^j\right)^2 \frac{\partial \gamma^j}{\partial y^j}}{1-d^j \frac{i}{1+r} \frac{\partial \gamma^j}{\partial y^j}}\right) \left(\sum_{j=\ell,h} \pi^j \left(\gamma^j\right)^2 \frac{\partial \widetilde{d}^j}{\partial q^j} \frac{1}{1-d^j \frac{i}{1+r} \frac{\partial \gamma^j}{\partial y^j}}\right) \\
= \; &
\end{aligned}$$

$$
\begin{aligned}
&= \frac{(1+r)\,\lambda\alpha^{h\ell}}{\mu}\pi^{\ell}\gamma^{\ell}\frac{\partial\widetilde{d}^{\ell}/\partial q^{\ell}}{1-d^{\ell}\frac{i}{1+r}\frac{\partial\gamma^{\ell}}{\partial y^{\ell}}}\left(\gamma^{h\ell}-\gamma^{\ell}\right)d^{h\ell} \\
&\quad+\frac{(1+r)\,\lambda\alpha^{h\ell}}{\mu}\pi^{h}\gamma^{h}\frac{\partial\widetilde{d}^{h}/\partial q^{h}}{1-d^{h}\frac{i}{1+r}\frac{\partial\gamma^{h}}{\partial y^{h}}}\left[\left(\gamma^{h\ell}-\gamma^{h}\right)d^{h\ell}-\frac{1-d^{h\ell}\frac{i}{1+r}\frac{\partial\gamma^{h\ell}}{\partial y^{\ell}}}{1-d^{\ell}\frac{i}{1+r}\frac{\partial\gamma^{\ell}}{\partial y^{\ell}}}\left(\gamma^{\ell}-\gamma^{h}\right)d^{\ell}\right] \\
&\quad+\frac{g-r}{1+g}\frac{\pi^{\ell}\pi^{h}\frac{\partial\gamma^{\ell}}{\partial y^{\ell}}\left(d^{\ell}\right)^{2}\gamma^{h}\frac{\partial\widetilde{d}^{h}}{\partial q^{h}}}{\left(1-d^{\ell}\frac{i}{1+r}\frac{\partial\gamma^{\ell}}{\partial y^{\ell}}\right)\left(1-d^{h}\frac{i}{1+r}\frac{\partial\gamma^{h}}{\partial y^{h}}\right)}\left(\gamma^{\ell}-\gamma^{h}\right) \\
&\quad-\frac{g-r}{1+g}\frac{\pi^{\ell}\pi^{h}\frac{\partial\gamma^{h}}{\partial y^{h}}\left(d^{h}\right)^{2}\gamma^{\ell}\frac{\partial\widetilde{d}^{\ell}}{\partial q^{\ell}}}{\left(1-d^{\ell}\frac{i}{1+r}\frac{\partial\gamma^{\ell}}{\partial y^{\ell}}\right)\left(1-d^{h}\frac{i}{1+r}\frac{\partial\gamma^{h}}{\partial y^{h}}\right)}\left(\gamma^{\ell}-\gamma^{h}\right),
\end{aligned}
$$

leading to:

$$
\begin{aligned}
\Delta\tau &= \frac{(1+r)\,\lambda\alpha^{h\ell}}{\mu}\frac{\pi^{\ell}\gamma^{\ell}\partial\widetilde{d}^{\ell}/\partial q^{\ell}}{1-d^{\ell}\frac{i}{1+r}\frac{\partial\gamma^{\ell}}{\partial y^{\ell}}}\left(\gamma^{h\ell}-\gamma^{\ell}\right)d^{h\ell} \\
&\quad+\frac{(1+r)\,\lambda\alpha^{h\ell}}{\mu}\frac{\pi^{h}\gamma^{h}\partial\widetilde{d}^{h}/\partial q^{h}}{1-d^{h}\frac{i}{1+r}\frac{\partial\gamma^{h}}{\partial y^{h}}}\left[\left(\gamma^{h\ell}-\gamma^{h}\right)d^{h\ell}-\frac{1-d^{h\ell}\frac{i}{1+r}\frac{\partial\gamma^{h\ell}}{\partial y^{\ell}}}{1-d^{\ell}\frac{i}{1+r}\frac{\partial\gamma^{\ell}}{\partial y^{\ell}}}\left(\gamma^{\ell}-\gamma^{h}\right)d^{\ell}\right] \\
&\quad+\frac{g-r}{1+g}\frac{\left(\gamma^{\ell}-\gamma^{h}\right)\pi^{\ell}\pi^{h}}{\left(1-d^{\ell}\frac{i}{1+r}\frac{\partial\gamma^{\ell}}{\partial y^{\ell}}\right)\left(1-d^{h}\frac{i}{1+r}\frac{\partial\gamma^{h}}{\partial y^{h}}\right)}\left[\frac{\partial\gamma^{\ell}}{\partial y^{\ell}}\left(d^{\ell}\right)^{2}\gamma^{h}\frac{\partial\widetilde{d}^{h}}{\partial q^{h}}-\frac{\partial\gamma^{h}}{\partial y^{h}}\left(d^{h}\right)^{2}\gamma^{\ell}\frac{\partial\widetilde{d}^{\ell}}{\partial q^{\ell}}\right],
\end{aligned}
$$

and finally:

$$
\begin{aligned}
\tau &= \frac{(1+r)\,\lambda\alpha^{h\ell}}{\Delta\mu}\frac{\pi^{\ell}\gamma^{\ell}\partial\widetilde{d}^{\ell}/\partial q^{\ell}}{1-d^{\ell}\frac{i}{1+r}\frac{\partial\gamma^{\ell}}{\partial y^{\ell}}}\left(\gamma^{h\ell}-\gamma^{\ell}\right)d^{h\ell} \\
&\quad+\frac{(1+r)\,\lambda\alpha^{h\ell}}{\Delta\mu}\frac{\pi^{h}\gamma^{h}\partial\widetilde{d}^{h}/\partial q^{h}}{1-d^{h}\frac{i}{1+r}\frac{\partial\gamma^{h}}{\partial y^{h}}}\left[\left(\gamma^{h\ell}-\gamma^{h}\right)d^{h\ell}-\frac{1-d^{h\ell}\frac{i}{1+r}\frac{\partial\gamma^{h\ell}}{\partial y^{\ell}}}{1-d^{\ell}\frac{i}{1+r}\frac{\partial\gamma^{\ell}}{\partial y^{\ell}}}\left(\gamma^{\ell}-\gamma^{h}\right)d^{\ell}\right] \\
&\quad+\frac{1}{\left(\gamma^{\ell}-\gamma^{h}\right)\frac{\partial\widetilde{d}^{h}}{\partial q^{h}}\frac{\partial\widetilde{d}^{\ell}}{\partial q^{\ell}}}\frac{g-r}{1+g}\left[\frac{\partial\gamma^{\ell}}{\partial y^{\ell}}\left(d^{\ell}\right)^{2}\gamma^{h}\frac{\partial\widetilde{d}^{h}}{\partial q^{h}}-\frac{\partial\gamma^{h}}{\partial y^{h}}\left(d^{h}\right)^{2}\gamma^{\ell}\frac{\partial\widetilde{d}^{\ell}}{\partial q^{\ell}}\right].
\end{aligned}
$$

Then we have:

$$
\begin{aligned}
&\Delta\left(\frac{g-r}{1+g}+i\right)\\
=\;&\\
&-\left(\sum_{j=\ell,h}\pi^j\frac{\partial\widetilde{d}^j}{\partial q^j}\frac{1}{1-d^j\frac{i}{1+r}\frac{\partial\gamma^j}{\partial y^j}}\right)\left[\frac{(1+r)\,\lambda\alpha^{h\ell}}{\mu}\left(\gamma^{h\ell}d^{h\ell}-\frac{1-d^{h\ell}\frac{i}{1+r}\frac{\partial\gamma^{h\ell}}{\partial y^\ell}}{1-d^\ell\frac{i}{1+r}\frac{\partial\gamma^\ell}{\partial y^\ell}}\gamma^\ell d^\ell\right)\right]\\
&-\left(\sum_{j=\ell,h}\pi^j\frac{\partial\widetilde{d}^j}{\partial q^j}\frac{1}{1-d^j\frac{i}{1+r}\frac{\partial\gamma^j}{\partial y^j}}\right)\frac{g-r}{1+g}\sum_{j=\ell,h}\frac{\pi^j\frac{\partial\gamma^j}{\partial y^j}d^j\gamma^jd^j}{1-d^j\frac{i}{1+r}\frac{\partial\gamma^j}{\partial y^j}}\\
&+\left(\sum_{j=\ell,h}\pi^j\gamma^j\frac{\partial\widetilde{d}^j}{\partial q^j}\frac{1}{1-d^j\frac{i}{1+r}\frac{\partial\gamma^j}{\partial y^j}}\right)\left[\frac{(1+r)\,\lambda\alpha^{h\ell}}{\mu}\left(d^{h\ell}-\frac{1-d^{h\ell}\frac{i}{1+r}\frac{\partial\gamma^{h\ell}}{\partial y^\ell}}{1-d^\ell\frac{i}{1+r}\frac{\partial\gamma^\ell}{\partial y^\ell}}d^\ell\right)\right]\\
&+\left(\sum_{j=\ell,h}\pi^j\gamma^j\frac{\partial\widetilde{d}^j}{\partial q^j}\frac{1}{1-d^j\frac{i}{1+r}\frac{\partial\gamma^j}{\partial y^j}}\right)\frac{g-r}{1+g}\sum_{j=\ell,h}\frac{\pi^j\left(d^j\right)^2\frac{\partial\gamma^j}{\partial y^j}}{1-d^j\frac{i}{1+r}\frac{\partial\gamma^j}{\partial y^j}}\\
=\;&\\
&\frac{(1+r)\,\lambda\alpha^{h\ell}}{\mu}\left(d^{h\ell}-\frac{1-d^{h\ell}\frac{i}{1+r}\frac{\partial\gamma^{h\ell}}{\partial y^\ell}}{1-d^\ell\frac{i}{1+r}\frac{\partial\gamma^\ell}{\partial y^\ell}}d^\ell\right)\left(\sum_{j=\ell,h}\pi^j\gamma^j\frac{\partial\widetilde{d}^j}{\partial q^j}\frac{1}{1-d^j\frac{i}{1+r}\frac{\partial\gamma^j}{\partial y^j}}\right)\\
&-\frac{(1+r)\,\lambda\alpha^{h\ell}}{\mu}\left(\gamma^{h\ell}d^{h\ell}-\frac{1-d^{h\ell}\frac{i}{1+r}\frac{\partial\gamma^{h\ell}}{\partial y^\ell}}{1-d^\ell\frac{i}{1+r}\frac{\partial\gamma^\ell}{\partial y^\ell}}\gamma^\ell d^\ell\right)\left(\sum_{j=\ell,h}\pi^j\frac{\partial\widetilde{d}^j}{\partial q^j}\frac{1}{1-d^j\frac{i}{1+r}\frac{\partial\gamma^j}{\partial y^j}}\right)\\
&+\frac{g-r}{1+g}\left(\sum_{j=\ell,h}\pi^j\gamma^j\frac{\partial\widetilde{d}^j}{\partial q^j}\frac{1}{1-d^j\frac{i}{1+r}\frac{\partial\gamma^j}{\partial y^j}}\right)\left(\sum_{j=\ell,h}\frac{\pi^j\left(d^j\right)^2\frac{\partial\gamma^j}{\partial y^j}}{1-d^j\frac{i}{1+r}\frac{\partial\gamma^j}{\partial y^j}}\right)\\
&-\frac{g-r}{1+g}\left(\sum_{j=\ell,h}\pi^j\frac{\partial\widetilde{d}^j}{\partial q^j}\frac{1}{1-d^j\frac{i}{1+r}\frac{\partial\gamma^j}{\partial y^j}}\right)\left(\sum_{j=\ell,h}\frac{\pi^j\frac{\partial\gamma^j}{\partial y^j}d^j\gamma^jd^j}{1-d^j\frac{i}{1+r}\frac{\partial\gamma^j}{\partial y^j}}\right)\\
=\;&
\end{aligned}
$$

$$
\begin{aligned}
= \quad & \frac{(1+r)\,\lambda\alpha^{h\ell}}{\mu}\frac{\pi^{\ell}\partial\widetilde{d}^{\ell}/\partial q^{\ell}}{1-d^{\ell}\frac{i}{1+r}\frac{\partial\gamma^{\ell}}{\partial y^{\ell}}}\left(\gamma^{\ell}-\gamma^{h\ell}\right)d^{h\ell}\\
& +\frac{(1+r)\,\lambda\alpha^{h\ell}}{\mu}\frac{\pi^{h}\partial\widetilde{d}^{h}/\partial q^{h}}{1-d^{h}\frac{i}{1+r}\frac{\partial\gamma^{h}}{\partial y^{h}}}\left[\left(\gamma^{\ell}-\gamma^{h}\right)\frac{1-d^{h\ell}\frac{i}{1+r}\frac{\partial\gamma^{h\ell}}{\partial y^{\ell}}}{1-d^{\ell}\frac{i}{1+r}\frac{\partial\gamma^{\ell}}{\partial y^{\ell}}}d^{\ell}-\left(\gamma^{h\ell}-\gamma^{h}\right)d^{h\ell}\right]\\
& +\frac{g-r}{1+g}\frac{\pi^{\ell}\pi^{h}\frac{\partial\gamma^{h}}{\partial y^{h}}\left(d^{h}\right)^{2}\frac{\partial\widetilde{d}^{\ell}}{\partial q^{\ell}}}{\left(1-d^{h}\frac{i}{1+r}\frac{\partial\gamma^{h}}{\partial y^{h}}\right)\left(1-d^{\ell}\frac{i}{1+r}\frac{\partial\gamma^{\ell}}{\partial y^{\ell}}\right)}\left(\gamma^{\ell}-\gamma^{h}\right)\\
& -\frac{g-r}{1+g}\frac{\pi^{\ell}\pi^{h}\frac{\partial\gamma^{\ell}}{\partial y^{\ell}}\left(d^{\ell}\right)^{2}\frac{\partial\widetilde{d}^{h}}{\partial q^{h}}}{\left(1-d^{h}\frac{i}{1+r}\frac{\partial\gamma^{h}}{\partial y^{h}}\right)\left(1-d^{\ell}\frac{i}{1+r}\frac{\partial\gamma^{\ell}}{\partial y^{\ell}}\right)}\left(\gamma^{\ell}-\gamma^{h}\right)\\
= \quad & \frac{(1+r)\,\lambda\alpha^{h\ell}}{\mu}\frac{\pi^{\ell}\partial\widetilde{d}^{\ell}/\partial q^{\ell}}{1-d^{\ell}\frac{i}{1+r}\frac{\partial\gamma^{\ell}}{\partial y^{\ell}}}\left(\gamma^{\ell}-\gamma^{h\ell}\right)d^{h\ell}\\
& +\frac{(1+r)\,\lambda\alpha^{h\ell}}{\mu}\frac{\pi^{h}\partial\widetilde{d}^{h}/\partial q^{h}}{1-d^{h}\frac{i}{1+r}\frac{\partial\gamma^{h}}{\partial y^{h}}}\left[\left(\gamma^{\ell}-\gamma^{h}\right)\frac{1-d^{h\ell}\frac{i}{1+r}\frac{\partial\gamma^{h\ell}}{\partial y^{\ell}}}{1-d^{\ell}\frac{i}{1+r}\frac{\partial\gamma^{\ell}}{\partial y^{\ell}}}d^{\ell}-\left(\gamma^{h\ell}-\gamma^{h}\right)d^{h\ell}\right]\\
& +\frac{g-r}{1+g}\frac{\left(\gamma^{\ell}-\gamma^{h}\right)\pi^{\ell}\pi^{h}}{\left(1-d^{h}\frac{i}{1+r}\frac{\partial\gamma^{h}}{\partial y^{h}}\right)\left(1-d^{\ell}\frac{i}{1+r}\frac{\partial\gamma^{\ell}}{\partial y^{\ell}}\right)}\left[\frac{\partial\gamma^{h}}{\partial y^{h}}\left(d^{h}\right)^{2}\frac{\partial\widetilde{d}^{\ell}}{\partial q^{\ell}}-\frac{\partial\gamma^{\ell}}{\partial y^{\ell}}\left(d^{\ell}\right)^{2}\frac{\partial\widetilde{d}^{h}}{\partial q^{h}}\right],
\end{aligned}
$$

leading to:

$$
\begin{aligned}
& \frac{g-r}{1+g}+i\\
= \quad & \\
& \frac{(1+r)\,\lambda\alpha^{h\ell}}{\Delta\mu}\frac{\pi^{\ell}\partial\widetilde{d}^{\ell}/\partial q^{\ell}}{1-d^{\ell}\frac{i}{1+r}\frac{\partial\gamma^{\ell}}{\partial y^{\ell}}}\left(\gamma^{\ell}-\gamma^{h\ell}\right)d^{h\ell}\\
& +\frac{(1+r)\,\lambda\alpha^{h\ell}}{\Delta\mu}\frac{\pi^{h}\partial\widetilde{d}^{h}/\partial q^{h}}{1-d^{h}\frac{i}{1+r}\frac{\partial\gamma^{h}}{\partial y^{h}}}\left[\left(\gamma^{\ell}-\gamma^{h}\right)\frac{1-d^{h\ell}\frac{i}{1+r}\frac{\partial\gamma^{h\ell}}{\partial y^{\ell}}}{1-d^{\ell}\frac{i}{1+r}\frac{\partial\gamma^{\ell}}{\partial y^{\ell}}}d^{\ell}-\left(\gamma^{h\ell}-\gamma^{h}\right)d^{h\ell}\right]\\
& +\frac{1}{\left(\gamma^{\ell}-\gamma^{h}\right)\frac{\partial\widetilde{d}^{\ell}}{\partial q^{\ell}}\frac{\partial\widetilde{d}^{h}}{\partial q^{h}}}\frac{g-r}{1+g}\left[\frac{\partial\gamma^{h}}{\partial y^{h}}\left(d^{h}\right)^{2}\frac{\partial\widetilde{d}^{\ell}}{\partial q^{\ell}}-\frac{\partial\gamma^{\ell}}{\partial y^{\ell}}\left(d^{\ell}\right)^{2}\frac{\partial\widetilde{d}^{h}}{\partial q^{h}}\right],
\end{aligned}
$$

and therefore:

$$
\begin{aligned}
i &= \frac{(1+r)\,\lambda\alpha^{h\ell}}{\Delta\mu}\frac{\pi^{\ell}\partial\widetilde{d}^{\ell}/\partial q^{\ell}}{1-d^{\ell}\frac{i}{1+r}\frac{\partial\gamma^{\ell}}{\partial y^{\ell}}}\left(\gamma^{\ell}-\gamma^{h\ell}\right)d^{h\ell} \\
&\quad+\frac{(1+r)\,\lambda\alpha^{h\ell}}{\Delta\mu}\frac{\pi^{h}\partial\widetilde{d}^{h}/\partial q^{h}}{1-d^{h}\frac{i}{1+r}\frac{\partial\gamma^{h}}{\partial y^{h}}}\left[\left(\gamma^{\ell}-\gamma^{h}\right)\frac{1-d^{h\ell}\frac{i}{1+r}\frac{\partial\gamma^{h\ell}}{\partial y^{\ell}}}{1-d^{\ell}\frac{i}{1+r}\frac{\partial\gamma^{\ell}}{\partial y^{\ell}}}d^{\ell}-\left(\gamma^{h\ell}-\gamma^{h}\right)d^{h\ell}\right] \\
&\quad+\frac{g-r}{1+g}\left[\frac{\frac{\partial\gamma^{h}}{\partial y^{h}}\left(d^{h}\right)^{2}\frac{\partial\widetilde{d}^{\ell}}{\partial q^{\ell}}-\frac{\partial\gamma^{\ell}}{\partial y^{\ell}}\left(d^{\ell}\right)^{2}\frac{\partial\widetilde{d}^{h}}{\partial q^{h}}}{\left(\gamma^{\ell}-\gamma^{h}\right)\frac{\partial\widetilde{d}^{\ell}}{\partial q^{\ell}}\frac{\partial\widetilde{d}^{h}}{\partial q^{h}}}-1\right].
\end{aligned}
$$

Appendix B

Proof of Proposition 7: With $\gamma(\cdot) = \gamma(w, I)$ all the terms $\partial\gamma/\partial y$ in (53) vanish and one can rewrite (53) as

$$\left[\gamma^{h\ell} - \gamma^h + \left(\gamma^{h\ell} - \gamma^\ell\right) \frac{\pi^\ell}{\pi^h} \frac{\frac{\partial \widetilde{d}^\ell}{\partial q^\ell}}{\frac{\partial \widetilde{d}^h}{\partial q^h}}\right] d^{h\ell} - \left(\gamma^\ell - \gamma^h\right) d^\ell \leq 0. \tag{B1}$$

Notice that with $\gamma(w, I)$ being quasi-linear in income, the difference $\gamma^\ell - \gamma^{h\ell}$ is independent on I (the two functions $\gamma\left(w^\ell, I\right)$ and $\gamma\left(w^h, I\right)$ are parallel). Then consider the effect of a parallel marginal downward shift of the $\gamma\left(w^h, I\right)$ function, which is tantamount to saying that the dependence of $\gamma(\cdot)$ on skills is made stronger. Labelling by Ψ the quantity appearing on the left hand side of (B1), the case for the optimality of the Friedman rule is strengthened if the effect of the change is to lower Ψ. Denoting by $\varepsilon_{d,q}^{h\ell} \equiv \left(\partial d^{h\ell}/\partial q^{h\ell}\right)\left(q^{h\ell}/d^{h\ell}\right) < 0$ the mimicker's elasticity of second period consumption with respect to the intertemporal price $q^{h\ell}$, we have:

$$\begin{aligned}
d\Psi &= -d^{h\ell}\left[\frac{\pi^\ell}{\pi^h}\frac{\frac{\partial \widetilde{d}^\ell}{\partial q^\ell}}{\frac{\partial \widetilde{d}^h}{\partial q^h}} - \left(\gamma^{h\ell} - \gamma^\ell\right)\frac{\pi^\ell}{\pi^h}\frac{\frac{\partial \widetilde{d}^\ell}{\partial q^\ell}\frac{\partial^2 \widetilde{d}^h}{\partial q^h \partial q^h}\frac{\partial q^h}{\partial \gamma^h}}{\left(\frac{\partial \widetilde{d}^h}{\partial q^h}\right)^2}\right] \\
&\quad - \left[\gamma^{h\ell} - \gamma^h + \left(\gamma^{h\ell} - \gamma^\ell\right)\frac{\pi^\ell}{\pi^h}\frac{\frac{\partial \widetilde{d}^\ell}{\partial q^\ell}}{\frac{\partial \widetilde{d}^h}{\partial q^h}}\right]\frac{\partial d^{h\ell}}{\partial q^{h\ell}}\frac{\partial q^{h\ell}}{\partial \gamma^{h\ell}} - d^\ell \\
&= -d^{h\ell}\left[\frac{\pi^\ell}{\pi^h}\frac{\frac{\partial \widetilde{d}^\ell}{\partial q^\ell}}{\frac{\partial \widetilde{d}^h}{\partial q^h}} - \left(\gamma^{h\ell} - \gamma^\ell\right)\frac{\pi^\ell}{\pi^h}\frac{\frac{\partial \widetilde{d}^\ell}{\partial q^\ell}\frac{\partial^2 \widetilde{d}^h}{\partial q^h \partial q^h}\frac{i}{1+r}}{\left(\frac{\partial \widetilde{d}^h}{\partial q^h}\right)^2}\right] \\
&\quad - \left[\gamma^{h\ell} - \gamma^h + \left(\gamma^{h\ell} - \gamma^\ell\right)\frac{\pi^\ell}{\pi^h}\frac{\frac{\partial \widetilde{d}^\ell}{\partial q^\ell}}{\frac{\partial \widetilde{d}^h}{\partial q^h}}\right]\frac{\partial d^{h\ell}}{\partial q^{h\ell}}\frac{i}{1+r} - d^\ell \\
&=
\end{aligned}$$

$$
\begin{aligned}
&= -d^{h\ell}\left[\frac{\pi^{\ell}}{\pi^{h}}\frac{\frac{\partial \widetilde{d}^{\ell}}{\partial q^{\ell}}}{\frac{\partial \widetilde{d}^{h}}{\partial q^{h}}} - \left(\gamma^{h\ell}-\gamma^{\ell}\right)\frac{\pi^{\ell}}{\pi^{h}}\frac{\frac{\partial \widetilde{d}^{\ell}}{\partial q^{\ell}}\frac{\partial^2 \widetilde{d}^{h}}{\partial q^{h}\partial q^{h}}\frac{i}{1+r}}{\left(\frac{\partial \widetilde{d}^{h}}{\partial q^{h}}\right)^2}\right] \\
&\quad - \left[\gamma^{h\ell}-\gamma^{h}+\left(\gamma^{h\ell}-\gamma^{\ell}\right)\frac{\pi^{\ell}}{\pi^{h}}\frac{\frac{\partial \widetilde{d}^{\ell}}{\partial q^{\ell}}}{\frac{\partial \widetilde{d}^{h}}{\partial q^{h}}}\right]\varepsilon^{h\ell}_{d,q}\frac{d^{h\ell}}{q^{h\ell}}\frac{i}{1+r} - d^{\ell} \\
&= -d^{h\ell}\left[\frac{\pi^{\ell}}{\pi^{h}}\frac{\frac{\partial \widetilde{d}^{\ell}}{\partial q^{\ell}}}{\frac{\partial \widetilde{d}^{h}}{\partial q^{h}}} - \left(\gamma^{h\ell}-\gamma^{\ell}\right)\frac{\pi^{\ell}}{\pi^{h}}\frac{\frac{\partial \widetilde{d}^{\ell}}{\partial q^{\ell}}\frac{\partial^2 \widetilde{d}^{h}}{\partial q^{h}\partial q^{h}}\frac{i}{1+r}}{\left(\frac{\partial \widetilde{d}^{h}}{\partial q^{h}}\right)^2}\right] \\
&\quad - \left[\gamma^{h\ell}-\gamma^{h}+\left(\gamma^{h\ell}-\gamma^{\ell}\right)\frac{\pi^{\ell}}{\pi^{h}}\frac{\frac{\partial \widetilde{d}^{\ell}}{\partial q^{\ell}}}{\frac{\partial \widetilde{d}^{h}}{\partial q^{h}}}\right]\varepsilon^{h\ell}_{d,q}d^{h\ell}\frac{i}{1+\tau+\gamma^{h\ell}i} - d^{\ell} \\
&= -d^{h\ell}\left\{\underbrace{\frac{\pi^{\ell}}{\pi^{h}}\frac{\frac{\partial \widetilde{d}^{\ell}}{\partial q^{\ell}}}{\frac{\partial \widetilde{d}^{h}}{\partial q^{h}}} - \left(\gamma^{h\ell}-\gamma^{\ell}\right)\frac{\pi^{\ell}}{\pi^{h}}\frac{\frac{\partial \widetilde{d}^{\ell}}{\partial q^{\ell}}\frac{\partial^2 \widetilde{d}^{h}}{\partial q^{h}\partial q^{h}}\frac{i}{1+r}}{\left(\frac{\partial \widetilde{d}^{h}}{\partial q^{h}}\right)^2}}_{>0}\right\} - d^{\ell} \\
&\quad \underbrace{-d^{h\ell}\left[\gamma^{h\ell}-\gamma^{h}+\left(\gamma^{h\ell}-\gamma^{\ell}\right)\frac{\pi^{\ell}}{\pi^{h}}\frac{\frac{\partial \widetilde{d}^{\ell}}{\partial q^{\ell}}}{\frac{\partial \widetilde{d}^{h}}{\partial q^{h}}}\right]\frac{i}{1+\tau+\gamma^{h\ell}i}}_{\geq 0}\varepsilon^{h\ell}_{d,q}.
\end{aligned}
$$

Thus, unless $\mid \varepsilon^{h\ell}_{d,q} \mid$ is very large, the effect of the postulated change is to strengthen the case for the optimality of the Friedman rule. A similar result, but this time requiring $\varepsilon^{\ell}_{d,q}$ to be not too large in absolute value, can be obtained by considering the effect on Ψ of a parallel marginal upward shift of the $\gamma\left(w^{\ell}, I\right)$ function. Therefore, starting from a situation where γ depends only on gross income, letting it also depend on skills strengthens the case for the optimality of the Friedman rule when the elasticity of demand for second period consumption with respect to the intertemporal price q is not too large. Given that when γ depends only on income a necessary and sufficient condition for the optimality of the Friedman rule is that labor supply and second period consumption are not Hicksian substitutes, Hicksian non-substitutability between labor supply and second period consumption becomes a sufficient but no longer necessary condition for the Friedman rule to be optimal.

Appendix C

Proof of Proposition 8: Rewrite (53) as

$$\left[\gamma^{h\ell}-\gamma^{h}+\left(\gamma^{h\ell}-\gamma^{\ell}\right)\frac{\pi^{\ell}}{\pi^{h}}\frac{\frac{\partial\widetilde{d}^{\ell}}{\partial q^{\ell}}}{\frac{\partial\widetilde{d}^{h}}{\partial q^{h}}}\frac{1-d^{h}\frac{i}{1+r}\frac{\partial\gamma^{h}}{\partial y^{h}}}{1-d^{\ell}\frac{i}{1+r}\frac{\partial\gamma^{\ell}}{\partial y^{\ell}}}\right]d^{h\ell}-\frac{1-d^{h\ell}\frac{i}{1+r}\frac{\partial\gamma^{h\ell}}{\partial y^{\ell}}}{1-d^{\ell}\frac{i}{1+r}\frac{\partial\gamma^{\ell}}{\partial y^{\ell}}}\left(\gamma^{\ell}-\gamma^{h}\right)d^{\ell}\leq 0. \tag{C1}$$

With $\gamma(\cdot)=\gamma(w,y)$ being quasi-linear in income, the difference $\gamma^{\ell}-\gamma^{h\ell}$ is independent on y (the two functions $\gamma\left(w^{\ell},y\right)$ and $\gamma\left(w^{h},y\right)$ are parallel) and $\partial\gamma^{h\ell}/\partial y^{\ell}=\partial\gamma^{\ell}/\partial y^{\ell}\equiv\partial\gamma/\partial y$. Then consider the effect of a parallel marginal downward shift of the $\gamma\left(w^{h},y\right)$ function, which is tantamount to saying that the dependence of $\gamma(\cdot)$ on skills is made stronger. Labelling by Ω the quantity appearing on the left hand side of (C1), the case for the optimality of the Friedman rule is strengthened if the effect of the postulated change is to lower Ω. Denoting by $\varepsilon_{d,q}^{h\ell}\equiv\left(\partial d^{h\ell}/\partial q^{h\ell}\right)\left(q^{h\ell}/d^{h\ell}\right)<0$ the mimicker's elasticity of second period consumption with respect to the intertemporal price $q^{h\ell}$, we have:

$$\begin{aligned}
d\Omega &= -d^{h\ell}\frac{\pi^{\ell}}{\pi^{h}}\frac{\frac{\partial\widetilde{d}^{\ell}}{\partial q^{\ell}}}{\frac{\partial\widetilde{d}^{h}}{\partial q^{h}}}\frac{1-d^{h}\frac{i}{1+r}\frac{\partial\gamma^{h}}{\partial y^{h}}}{1-d^{\ell}\frac{i}{1+r}\frac{\partial\gamma^{\ell}}{\partial y^{\ell}}} \\
&\quad +d^{h\ell}\left(\gamma^{h\ell}-\gamma^{\ell}\right)\frac{\pi^{\ell}}{\pi^{h}}\frac{\frac{\partial\widetilde{d}^{\ell}}{\partial q^{\ell}}}{1-d^{\ell}\frac{i}{1+r}\frac{\partial\gamma^{\ell}}{\partial y^{\ell}}}\frac{\left[\frac{\partial^{2}\widetilde{d}^{h}}{\partial q^{h}\partial q^{h}}\frac{\partial q^{h}}{\partial\gamma^{h}}\left(1-d^{h}\frac{i}{1+r}\frac{\partial\gamma^{h}}{\partial y^{h}}\right)+\frac{\partial\widetilde{d}^{h}}{\partial q^{h}}\frac{i}{1+r}\frac{\partial\gamma^{h}}{\partial y^{h}}\frac{\partial d^{h}}{\partial q^{h}}\frac{\partial q^{h}}{\partial\gamma^{h}}\right]}{\left(\frac{\partial\widetilde{d}^{h}}{\partial q^{h}}\right)^{2}} \\
&\quad -\left[\gamma^{h\ell}-\gamma^{h}+\left(\gamma^{h\ell}-\gamma^{\ell}\right)\frac{\pi^{\ell}}{\pi^{h}}\frac{\frac{\partial\widetilde{d}^{\ell}}{\partial q^{\ell}}}{\frac{\partial\widetilde{d}^{h}}{\partial q^{h}}}\frac{1-d^{h}\frac{i}{1+r}\frac{\partial\gamma^{h}}{\partial y^{h}}}{1-d^{\ell}\frac{i}{1+r}\frac{\partial\gamma^{\ell}}{\partial y^{\ell}}}\right]\frac{\partial d^{h\ell}}{\partial q^{h\ell}}\frac{\partial q^{h\ell}}{\partial\gamma^{h\ell}} \\
&\quad -d^{\ell}\left[\frac{1-d^{h\ell}\frac{i}{1+r}\frac{\partial\gamma}{\partial y}}{1-d^{\ell}\frac{i}{1+r}\frac{\partial\gamma}{\partial y}}+\left(\gamma^{\ell}-\gamma^{h}\right)\frac{\frac{i}{1+r}\frac{\partial\gamma}{\partial y}\frac{\partial d^{h\ell}}{\partial q^{h\ell}}\frac{\partial q^{h\ell}}{\partial\gamma^{h\ell}}}{1-d^{\ell}\frac{i}{1+r}\frac{\partial\gamma}{\partial y}}\right] \\
&=
\end{aligned}$$

$$
\begin{aligned}
= \quad & -d^{h\ell}\frac{\pi^\ell}{\pi^h}\frac{\frac{\partial\widetilde{d}^\ell}{\partial q^\ell}}{\frac{\partial\widetilde{d}^h}{\partial q^h}}\frac{1-d^h\frac{i}{1+r}\frac{\partial\gamma^h}{\partial y^h}}{1-d^\ell\frac{i}{1+r}\frac{\partial\gamma^\ell}{\partial y^\ell}} \\
& +d^{h\ell}\left(\gamma^{h\ell}-\gamma^\ell\right)\frac{\pi^\ell}{\pi^h}\frac{\frac{\partial\widetilde{d}^\ell}{\partial q^\ell}}{1-d^\ell\frac{i}{1+r}\frac{\partial\gamma^\ell}{\partial y^\ell}}\frac{\left[\frac{\partial^2\widetilde{d}^h}{\partial q^h\partial q^h}\frac{i}{1+r}\left(1-d^h\frac{i}{1+r}\frac{\partial\gamma^h}{\partial y^h}\right)+\frac{\partial\widetilde{d}^h}{\partial q^h}\left(\frac{i}{1+r}\right)^2\frac{\partial\gamma^h}{\partial y^h}\frac{\partial d^h}{\partial q^h}\right]}{\left(\frac{\partial\widetilde{d}^h}{\partial q^h}\right)^2} \\
& -\left[\gamma^{h\ell}-\gamma^h+\left(\gamma^{h\ell}-\gamma^\ell\right)\frac{\pi^\ell}{\pi^h}\frac{\frac{\partial\widetilde{d}^\ell}{\partial q^\ell}}{\frac{\partial\widetilde{d}^h}{\partial q^h}}\frac{1-d^h\frac{i}{1+r}\frac{\partial\gamma^h}{\partial y^h}}{1-d^\ell\frac{i}{1+r}\frac{\partial\gamma^\ell}{\partial y^\ell}}\right]\frac{\partial d^{h\ell}}{\partial q^{h\ell}}\frac{i}{1+r} \\
& -d^\ell\left[\frac{1-d^{h\ell}\frac{i}{1+r}\frac{\partial\gamma}{\partial y}}{1-d^\ell\frac{i}{1+r}\frac{\partial\gamma}{\partial y}}+\left(\gamma^\ell-\gamma^h\right)\frac{\left(\frac{i}{1+r}\right)^2\frac{\partial\gamma}{\partial y}\frac{\partial d^{h\ell}}{\partial q^{h\ell}}}{1-d^\ell\frac{i}{1+r}\frac{\partial\gamma}{\partial y}}\right] \\
= \quad & -d^{h\ell}\frac{\pi^\ell}{\pi^h}\frac{\frac{\partial\widetilde{d}^\ell}{\partial q^\ell}}{\frac{\partial\widetilde{d}^h}{\partial q^h}}\frac{1-d^h\frac{i}{1+r}\frac{\partial\gamma^h}{\partial y^h}}{1-d^\ell\frac{i}{1+r}\frac{\partial\gamma^\ell}{\partial y^\ell}} \\
& +d^{h\ell}\left(\gamma^{h\ell}-\gamma^\ell\right)\frac{\pi^\ell}{\pi^h}\frac{\frac{\partial\widetilde{d}^\ell}{\partial q^\ell}\frac{i}{1+r}}{1-d^\ell\frac{i}{1+r}\frac{\partial\gamma^\ell}{\partial y^\ell}}\frac{\left[\frac{\partial^2\widetilde{d}^h}{\partial q^h\partial q^h}\left(1-d^h\frac{i}{1+r}\frac{\partial\gamma^h}{\partial y^h}\right)+\frac{\partial\widetilde{d}^h}{\partial q^h}\frac{i}{1+r}\frac{\partial\gamma^h}{\partial y^h}\frac{\partial d^h}{\partial q^h}\right]}{\left(\frac{\partial\widetilde{d}^h}{\partial q^h}\right)^2} \\
& -\left[\gamma^{h\ell}-\gamma^h+\left(\gamma^{h\ell}-\gamma^\ell\right)\frac{\pi^\ell}{\pi^h}\frac{\frac{\partial\widetilde{d}^\ell}{\partial q^\ell}}{\frac{\partial\widetilde{d}^h}{\partial q^h}}\frac{1-d^h\frac{i}{1+r}\frac{\partial\gamma^h}{\partial y^h}}{1-d^\ell\frac{i}{1+r}\frac{\partial\gamma^\ell}{\partial y^\ell}}\right]\varepsilon^{h\ell}_{d,q}\frac{d^{h\ell}}{q^{h\ell}}\frac{i}{1+r} \\
& -d^\ell\left[\frac{1-d^{h\ell}\frac{i}{1+r}\frac{\partial\gamma}{\partial y}}{1-d^\ell\frac{i}{1+r}\frac{\partial\gamma}{\partial y}}+\left(\gamma^\ell-\gamma^h\right)\frac{\left(\frac{i}{1+r}\right)^2\frac{\partial\gamma}{\partial y}\varepsilon^{h\ell}_{d,q}\frac{d^{h\ell}}{q^{h\ell}}}{1-d^\ell\frac{i}{1+r}\frac{\partial\gamma}{\partial y}}\right] \\
= \quad & -d^{h\ell}\frac{\pi^\ell}{\pi^h}\frac{\frac{\partial\widetilde{d}^\ell}{\partial q^\ell}}{\frac{\partial\widetilde{d}^h}{\partial q^h}}\frac{1-d^h\frac{i}{1+r}\frac{\partial\gamma^h}{\partial y^h}}{1-d^\ell\frac{i}{1+r}\frac{\partial\gamma^\ell}{\partial y^\ell}} \\
& +d^{h\ell}\left(\gamma^{h\ell}-\gamma^\ell\right)\frac{\pi^\ell}{\pi^h}\frac{\frac{\partial\widetilde{d}^\ell}{\partial q^\ell}\frac{i}{1+r}}{1-d^\ell\frac{i}{1+r}\frac{\partial\gamma^\ell}{\partial y^\ell}}\frac{\left[\frac{\partial^2\widetilde{d}^h}{\partial q^h\partial q^h}\left(1-d^h\frac{i}{1+r}\frac{\partial\gamma^h}{\partial y^h}\right)+\frac{\partial\widetilde{d}^h}{\partial q^h}\frac{i}{1+r}\frac{\partial\gamma^h}{\partial y^h}\frac{\partial d^h}{\partial q^h}\right]}{\left(\frac{\partial\widetilde{d}^h}{\partial q^h}\right)^2} \\
& -\left[\gamma^{h\ell}-\gamma^h+\left(\gamma^{h\ell}-\gamma^\ell\right)\frac{\pi^\ell}{\pi^h}\frac{\frac{\partial\widetilde{d}^\ell}{\partial q^\ell}}{\frac{\partial\widetilde{d}^h}{\partial q^h}}\frac{1-d^h\frac{i}{1+r}\frac{\partial\gamma^h}{\partial y^h}}{1-d^\ell\frac{i}{1+r}\frac{\partial\gamma^\ell}{\partial y^\ell}}\right]\varepsilon^{h\ell}_{d,q}d^{h\ell}\frac{i}{1+\tau+\gamma^{h\ell}i} \\
& -d^\ell\left[\frac{1-d^{h\ell}\frac{i}{1+r}\frac{\partial\gamma}{\partial y}}{1-d^\ell\frac{i}{1+r}\frac{\partial\gamma}{\partial y}}+\left(\gamma^\ell-\gamma^h\right)\frac{\frac{i}{1+r}\frac{\partial\gamma}{\partial y}\varepsilon^{h\ell}_{d,q}d^{h\ell}\frac{i}{1+\tau+\gamma^{h\ell}i}}{1-d^\ell\frac{i}{1+r}\frac{\partial\gamma}{\partial y}}\right];
\end{aligned}
$$

and finally:

$$
\begin{aligned}
d\Omega \quad = \quad & \underbrace{-d^{h\ell}\frac{\pi^{\ell}}{\pi^{h}}\frac{\frac{\partial\widetilde{d}^{\ell}}{\partial q^{\ell}}}{\frac{\partial\widetilde{d}^{h}}{\partial q^{h}}}\frac{1-d^{h}\frac{i}{1+r}\frac{\partial\gamma^{h}}{\partial y^{h}}}{1-d^{\ell}\frac{i}{1+r}\frac{\partial\gamma^{\ell}}{\partial y^{\ell}}}}_{<0}\quad\underbrace{-d^{\ell}\frac{1-d^{h\ell}\frac{i}{1+r}\frac{\partial\gamma}{\partial y}}{1-d^{\ell}\frac{i}{1+r}\frac{\partial\gamma}{\partial y}}}_{<0} \\
& \underbrace{+d^{h\ell}\left(\gamma^{h\ell}-\gamma^{\ell}\right)\frac{\pi^{\ell}}{\pi^{h}}\frac{\frac{\partial\widetilde{d}^{\ell}}{\partial q^{\ell}}\frac{i}{1+r}}{1-d^{\ell}\frac{i}{1+r}\frac{\partial\gamma^{\ell}}{\partial y^{\ell}}}\frac{\left[\frac{\partial^{2}\widetilde{d}^{h}}{\partial q^{h}\partial q^{h}}\left(1-d^{h}\frac{i}{1+r}\frac{\partial\gamma^{h}}{\partial y^{h}}\right)+\frac{\partial\widetilde{d}^{h}}{\partial q^{h}}\frac{i}{1+r}\frac{\partial\gamma^{h}}{\partial y^{h}}\frac{\partial d^{h}}{\partial q^{h}}\right]}{\left(\frac{\partial\widetilde{d}^{h}}{\partial q^{h}}\right)^{2}}}_{\leq 0} \\
& \underbrace{-d^{h\ell}\left(\gamma^{\ell}-\gamma^{h}\right)\frac{d^{\ell}\frac{i}{1+r}\frac{\partial\gamma}{\partial y}}{1-d^{\ell}\frac{i}{1+r}\frac{\partial\gamma}{\partial y}}\frac{i}{1+\tau+\gamma^{h\ell}i}\varepsilon_{d,q}^{h\ell}}_{\leq 0} \\
& \underbrace{-d^{h\ell}\left[\gamma^{h\ell}-\gamma^{h}+\left(\gamma^{h\ell}-\gamma^{\ell}\right)\frac{\pi^{\ell}}{\pi^{h}}\frac{\frac{\partial\widetilde{d}^{\ell}}{\partial q^{\ell}}}{\frac{\partial\widetilde{d}^{h}}{\partial q^{h}}}\frac{1-d^{h}\frac{i}{1+r}\frac{\partial\gamma^{h}}{\partial y^{h}}}{1-d^{\ell}\frac{i}{1+r}\frac{\partial\gamma^{\ell}}{\partial y^{\ell}}}\right]\frac{i}{1+\tau+\gamma^{h\ell}i}\varepsilon_{d,q}^{h\ell}}_{\geq 0}.
\end{aligned}
$$

The only term which may be non-negative in the expression above is the last term. However, when $\mid \partial\gamma/\partial y \mid$ is large enough, the overall sign of the terms depending on $\varepsilon_{d,q}^{h\ell}$ becomes non-positive, leading to $d\Omega < 0$ and strengthening the case for the optimality of the Friedman rule. On the other hand, when $\mid \partial\gamma/\partial y \mid$ is small, the overall sign of the terms depending on $\varepsilon_{d,q}^{h\ell}$ becomes non-negative and then $d\Omega$ will be negative only provided that $\varepsilon_{d,q}^{h\ell}$ is not too large in absolute value.

Similar results can be obtained by considering the effect on Ω of a parallel marginal upward shift of the $\gamma\left(w^{\ell}, y\right)$ function. Therefore, starting from a situation where γ depends only on aggregate disposable income, letting it also depend on skills strengthens the case for the optimality of the Friedman rule when $\mid \partial\gamma/\partial y \mid$ is large enough; for low values of $\mid \partial\gamma/\partial y \mid$ whether or not the case for the optimality of the Friedman rule is strengthened depends on the magnitude of the elasticity of demand for second period consumption with respect to the intertemporal price q: if $\mid \varepsilon_{d,q} \mid$ is not too large the case for the optimality of the Friedman rule is strengthened.

Appendix D

Proof of equations (56)–(57): Substitute for τ and i from (49) and (50) in the expression for t^j in the text and collect terms. We have:

$$
\begin{aligned}
& t^j (1+r) \\
= & \\
& \frac{(1+r)\lambda\alpha^{h\ell}}{\Delta\mu} \frac{\pi^\ell \gamma^\ell \partial \widetilde{d}^\ell / \partial q^\ell}{1 - d^\ell \frac{i}{1+r}\frac{\partial \gamma^\ell}{\partial y^\ell}} \left(\gamma^{h\ell} - \gamma^\ell\right) d^{h\ell} \\
& + \frac{(1+r)\lambda\alpha^{h\ell}}{\Delta\mu} \frac{\pi^h \gamma^h \partial \widetilde{d}^h / \partial q^h}{1 - d^h \frac{i}{1+r}\frac{\partial \gamma^h}{\partial y^h}} \left[\left(\gamma^{h\ell} - \gamma^h\right) d^{h\ell} - \frac{1 - d^{h\ell}\frac{i}{1+r}\frac{\partial \gamma^{h\ell}}{\partial y^\ell}}{1 - d^\ell \frac{i}{1+r}\frac{\partial \gamma^\ell}{\partial y^\ell}} \left(\gamma^\ell - \gamma^h\right) d^\ell\right] \\
& + \frac{1}{\left(\gamma^\ell - \gamma^h\right)\frac{\partial \widetilde{d}^h}{\partial q^h}\frac{\partial \widetilde{d}^\ell}{\partial q^\ell}} \frac{g-r}{1+g} \left[\frac{\partial \gamma^\ell}{\partial y^\ell}\left(d^\ell\right)^2 \gamma^h \frac{\partial \widetilde{d}^h}{\partial q^h} - \frac{\partial \gamma^h}{\partial y^h}\left(d^h\right)^2 \gamma^\ell \frac{\partial \widetilde{d}^\ell}{\partial q^\ell}\right] \\
& + \gamma^j \frac{(1+r)\lambda\alpha^{h\ell}}{\Delta\mu} \frac{\pi^\ell \partial \widetilde{d}^\ell / \partial q^\ell}{1 - d^\ell \frac{i}{1+r}\frac{\partial \gamma^\ell}{\partial y^\ell}} \left(\gamma^\ell - \gamma^{h\ell}\right) d^{h\ell} \\
& + \gamma^j \frac{(1+r)\lambda\alpha^{h\ell}}{\Delta\mu} \frac{\pi^h \partial \widetilde{d}^h / \partial q^h}{1 - d^h \frac{i}{1+r}\frac{\partial \gamma^h}{\partial y^h}} \left[\left(\gamma^\ell - \gamma^h\right) \frac{1 - d^{h\ell}\frac{i}{1+r}\frac{\partial \gamma^{h\ell}}{\partial y^\ell}}{1 - d^\ell \frac{i}{1+r}\frac{\partial \gamma^\ell}{\partial y^\ell}} d^\ell - \left(\gamma^{h\ell} - \gamma^h\right) d^{h\ell}\right] \\
& + \gamma^j \frac{g-r}{1+g} \left[\frac{\frac{\partial \gamma^h}{\partial y^h}\left(d^h\right)^2 \frac{\partial \widetilde{d}^\ell}{\partial q^\ell} - \frac{\partial \gamma^\ell}{\partial y^\ell}\left(d^\ell\right)^2 \frac{\partial \widetilde{d}^h}{\partial q^h}}{\left(\gamma^\ell - \gamma^h\right)\frac{\partial \widetilde{d}^\ell}{\partial q^\ell}\frac{\partial \widetilde{d}^h}{\partial q^h}} - 1\right].
\end{aligned}
$$

Setting $j = h, \ell$, dividing by $1+r$ and simplifying terms, the above is written as:

$$
\begin{aligned}
t^h \quad = \quad & \frac{\lambda\alpha^{h\ell}}{\Delta\mu} \left[\frac{\pi^\ell \gamma^\ell \partial \widetilde{d}^\ell / \partial q^\ell}{1 - d^\ell \frac{i}{1+r}\frac{\partial \gamma^\ell}{\partial y^\ell}} \left(\gamma^{h\ell} - \gamma^\ell\right) d^{h\ell} + \frac{\gamma^h \pi^\ell \partial \widetilde{d}^\ell / \partial q^\ell}{1 - d^\ell \frac{i}{1+r}\frac{\partial \gamma^\ell}{\partial y^\ell}} \left(\gamma^\ell - \gamma^{h\ell}\right) d^{h\ell}\right] \\
& - \frac{(g-r)\frac{\partial \gamma^h}{\partial y^h}\left(d^h\right)^2}{(1+g)(1+r)\frac{\partial \widetilde{d}^h}{\partial q^h}} - \gamma^h \frac{g-r}{(1+g)(1+r)};
\end{aligned}
$$

$$
\begin{aligned}
t^{\ell} \;=\; & \frac{\lambda\alpha^{h\ell}}{\Delta\mu}\frac{\pi^{h}\gamma^{h}\partial\widetilde{d}^{h}/\partial q^{h}}{1-d^{h}\frac{i}{1+r}\frac{\partial\gamma^{h}}{\partial y^{h}}}\left[\left(\gamma^{h\ell}-\gamma^{h}\right)d^{h\ell}-\frac{1-d^{h\ell}\frac{i}{1+r}\frac{\partial\gamma^{h\ell}}{\partial y^{\ell}}}{1-d^{\ell}\frac{i}{1+r}\frac{\partial\gamma^{\ell}}{\partial y^{\ell}}}\left(\gamma^{\ell}-\gamma^{h}\right)d^{\ell}\right] \\
& +\gamma^{\ell}\frac{\lambda\alpha^{h\ell}}{\Delta\mu}\frac{\pi^{h}\partial\widetilde{d}^{h}/\partial q^{h}}{1-d^{h}\frac{i}{1+r}\frac{\partial\gamma^{h}}{\partial y^{h}}}\left[\left(\gamma^{\ell}-\gamma^{h}\right)\frac{1-d^{h\ell}\frac{i}{1+r}\frac{\partial\gamma^{h\ell}}{\partial y^{\ell}}}{1-d^{\ell}\frac{i}{1+r}\frac{\partial\gamma^{\ell}}{\partial y^{\ell}}}d^{\ell}-\left(\gamma^{h\ell}-\gamma^{h}\right)d^{h\ell}\right] \\
& -\frac{(g-r)\frac{\partial\gamma^{\ell}}{\partial y^{\ell}}\left(d^{\ell}\right)^{2}}{(1+g)(1+r)\frac{\partial\widetilde{d}^{\ell}}{\partial q^{\ell}}}-\gamma^{\ell}\frac{g-r}{(1+g)(1+r)}.
\end{aligned}
$$

Collecting terms gives:

$$
\begin{aligned}
t^{h} \;=\; & \frac{\lambda\alpha^{h\ell}}{\Delta\mu}\frac{\pi^{\ell}\partial\widetilde{d}^{\ell}/\partial q^{\ell}}{1-d^{\ell}\frac{i}{1+r}\frac{\partial\gamma^{\ell}}{\partial y^{\ell}}}\left(\gamma^{h\ell}-\gamma^{\ell}\right)d^{h\ell}\left(\gamma^{\ell}-\gamma^{h}\right) \\
& -\frac{(g-r)\frac{\partial\gamma^{h}}{\partial y^{h}}\left(d^{h}\right)^{2}}{(1+g)(1+r)\frac{\partial\widetilde{d}^{h}}{\partial q^{h}}}-\gamma^{h}\frac{g-r}{(1+g)(1+r)}; \qquad \text{(D1)}
\end{aligned}
$$

$$
\begin{aligned}
t^{\ell} \;=\; & \frac{\lambda\alpha^{h\ell}}{\Delta\mu}\frac{\pi^{h}\partial\widetilde{d}^{h}/\partial q^{h}}{1-d^{h}\frac{i}{1+r}\frac{\partial\gamma^{h}}{\partial y^{h}}}\left[\left(\gamma^{\ell}-\gamma^{h}\right)\frac{1-d^{h\ell}\frac{i}{1+r}\frac{\partial\gamma^{h\ell}}{\partial y^{\ell}}}{1-d^{\ell}\frac{i}{1+r}\frac{\partial\gamma^{\ell}}{\partial y^{\ell}}}d^{\ell}-\left(\gamma^{h\ell}-\gamma^{h}\right)d^{h\ell}\right]\left(\gamma^{\ell}-\gamma^{h}\right) \\
& -\frac{(g-r)\frac{\partial\gamma^{\ell}}{\partial y^{\ell}}\left(d^{\ell}\right)^{2}}{(1+g)(1+r)\frac{\partial\widetilde{d}^{\ell}}{\partial q^{\ell}}}-\gamma^{\ell}\frac{g-r}{(1+g)(1+r)}. \qquad \text{(D2)}
\end{aligned}
$$

Finally, substituting $\Delta \equiv \pi^{\ell}\pi^{h}\frac{\partial\widetilde{d}^{\ell}}{\partial q^{\ell}}\frac{\partial\widetilde{d}^{h}}{\partial q^{h}}\frac{\left(\gamma^{\ell}-\gamma^{h}\right)^{2}}{\left(1-d^{h}\frac{i}{1+r}\frac{\partial\gamma^{h}}{\partial y^{h}}\right)\left(1-d^{\ell}\frac{i}{1+r}\frac{\partial\gamma^{\ell}}{\partial y^{\ell}}\right)}$ in (D1)-(D2) and simplifying terms delivers (56)-(57).

Appendix E

The effect of the reform on the self-selection constraint is given by

$$
\begin{aligned}
-\lambda dv^{h\ell} &= -\lambda\left(\frac{\partial v^{h\ell}}{\partial q^{h\ell}}dq^{h\ell} + \frac{\partial v^{h\ell}}{\partial y^{\ell}}dz^{\ell}\right) \\
&= -\lambda\left[\frac{\partial v^{h\ell}}{\partial q^{h\ell}}\left(\frac{d\tau+\gamma^{h\ell}}{1+r} + \frac{\partial q^{h\ell}}{\partial \gamma^{h\ell}}\frac{\partial \gamma^{h\ell}}{\partial y^{\ell}}dz^{\ell}\right) + \frac{\partial v^{h\ell}}{\partial y^{\ell}}dz^{\ell}\right] \\
&= \lambda\alpha^{h\ell}\left[d^{h\ell}\left(\frac{d\tau+\gamma^{h\ell}}{1+r} + \frac{i}{1+r}\frac{\partial \gamma^{h\ell}}{\partial y^{\ell}}dz^{\ell}\right) - dz^{\ell}\right] \\
&= \lambda\alpha^{h\ell}\left(d^{h\ell}\frac{\gamma^{h\ell}-\gamma^{h}}{1+r} + d^{h\ell}\frac{i}{1+r}\frac{\partial \gamma^{h\ell}}{\partial y^{\ell}}d^{\ell}\frac{1}{1+r}\frac{\gamma^{\ell}-\gamma^{h}}{1-d^{\ell}\frac{i}{1+r}\frac{\partial\gamma^{\ell}}{\partial y^{\ell}}} - d^{\ell}\frac{1}{1+r}\frac{\gamma^{\ell}-\gamma^{h}}{1-d^{\ell}\frac{i}{1+r}\frac{\partial\gamma^{\ell}}{\partial y^{\ell}}}\right) \\
&= \lambda\alpha^{h\ell}\left[d^{h\ell}\frac{\gamma^{h\ell}-\gamma^{h}}{1+r} - d^{\ell}\frac{1}{1+r}\frac{\gamma^{\ell}-\gamma^{h}}{1-d^{\ell}\frac{i}{1+r}\frac{\partial\gamma^{\ell}}{\partial y^{\ell}}}\left(1-d^{h\ell}\frac{i}{1+r}\frac{\partial\gamma^{h\ell}}{\partial y^{\ell}}\right)\right].
\end{aligned}
$$

With $\mu = -\eta$ (see Appendix A), the effects on the government's budget constraint and the money-injection constraint can be combined to obtain:

$$
\pi^{\ell}\left(\mu\frac{\tau}{1+r}\frac{\partial \widetilde{d}^{\ell}}{\partial q^{\ell}}dq^{\ell} - \eta\frac{\gamma^{\ell}i}{1+r}\frac{\partial \widetilde{d}^{\ell}}{\partial q^{\ell}}dq^{\ell}\right) = \mu\pi^{\ell}\left(\frac{\tau}{1+r}+\frac{\gamma^{\ell}i}{1+r}\right)\frac{\partial \widetilde{d}^{\ell}}{\partial q^{\ell}}\frac{1}{1-d^{\ell}\frac{i}{1+r}\frac{\partial\gamma^{\ell}}{\partial y^{\ell}}}\frac{\gamma^{\ell}-\gamma^{h}}{1+r}.
$$

If the pre-reform equilibrium was an optimum, the two effects should exactly offset. Taking into account that $t^{\ell} \equiv \left(\tau+\gamma^{\ell}i\right)/(1+r)$, this requires:

$$
\mu\pi^{\ell}t^{\ell}\frac{\partial \widetilde{d}^{\ell}}{\partial q^{\ell}}\frac{1}{1-d^{\ell}\frac{i}{1+r}\frac{\partial\gamma^{\ell}}{\partial y^{\ell}}}\frac{\gamma^{\ell}-\gamma^{h}}{1+r} = -\lambda\alpha^{h\ell}\left[d^{h\ell}\frac{\gamma^{h\ell}-\gamma^{h}}{1+r} - d^{\ell}\frac{1}{1+r}\frac{1-d^{h\ell}\frac{i}{1+r}\frac{\partial\gamma^{h\ell}}{\partial y^{\ell}}}{1-d^{\ell}\frac{i}{1+r}\frac{\partial\gamma^{\ell}}{\partial y^{\ell}}}\left(\gamma^{\ell}-\gamma^{h}\right)\right],
$$

or:

$$
t^{\ell}\left(\gamma^{\ell}-\gamma^{h}\right) = \frac{\lambda\alpha^{h\ell}\left(1-d^{\ell}\frac{i}{1+r}\frac{\partial\gamma^{\ell}}{\partial y^{\ell}}\right)}{\mu\pi^{\ell}\frac{\partial \widetilde{d}^{\ell}}{\partial q^{\ell}}}\left[d^{\ell}\frac{1-d^{h\ell}\frac{i}{1+r}\frac{\partial\gamma^{h\ell}}{\partial y^{\ell}}}{1-d^{\ell}\frac{i}{1+r}\frac{\partial\gamma^{\ell}}{\partial y^{\ell}}}\left(\gamma^{\ell}-\gamma^{h}\right) - d^{h\ell}\left(\gamma^{h\ell}-\gamma^{h}\right)\right].
$$

Dividing both sides by $\left(\gamma^{\ell}-\gamma^{h}\right)$ gives (57) for the case when $g = r$.

Appendix F: Observability of individual consumption levels

Let τ^j denote the tax rate levied on the second-period consumption of individuals of type j. This changes the expression for q^j in (42) to

$$q^j = \frac{1}{1+r} + \gamma^j \left(\frac{1}{1+g} - \frac{1}{1+r} \right) + \frac{\tau^j}{1+r} + \frac{\gamma^j \theta}{1+g}. \tag{F1}$$

It follows from this expression that if the fiscal authority changes τ^j by

$$d\tau^j = -\gamma^j \frac{1+r}{1+g} d\theta, \tag{F2}$$

$dq^j = 0$ whenever the monetary authority changes θ by $d\theta$. Moreover, observe again that the change in θ induces a change in b^j as well. As in Section 4 and Subsection 5.1, let the fiscal authority also change z^j according to $dz^j = -db^j$. This change ensures that $dy^j = dz^j + db^j = 0$. With $dy^j = dq^j = 0$ and no change in I^j, the instituted changes leave the utility of the h-types and the ℓ-types intact.

To check resource feasibility, observe first that with $\left(q^j, y^j, I^j\right)$ remaining unchanged, the j-type's demand for d does not change either. With $dd^j = 0$, the change in the government's net tax revenue is, from (35), while substituting τ^j for τ, $-db^j$ for dz^j, and the value of $d\tau^j$ from (F2)

$$dR = \pi^h db^h + \pi^\ell db^\ell - \frac{1}{1+g} \sum_{j=\ell,h} \pi^j \gamma^j d^j d\theta. \tag{F3}$$

As in the exercises in the text, the changes in θ and b^j must satisfy the money injection constraint equation (36). Given that $dd^j = 0$, we have

$$\sum_{j=\ell,h} \pi^j db^j = \frac{1}{1+g} \sum_{j=\ell,h} \pi^j \gamma^j d^j d\theta. \tag{F4}$$

Substituting from (F4) into (F3) results in $dR = 0$.

It remains for us to check the incentive compatibility constraints. To that end, consider the expression that one gets for q^{jk} when substitutes τ^k for τ in (A8). We have

$$q^{jk} = \frac{1}{1+r} + \gamma^{jk} \left(\frac{1}{1+g} - \frac{1}{1+r} \right) + \frac{\tau^k}{1+r} + \frac{\gamma^{jk} \theta}{1+g}. \tag{F5}$$

It then follows from (F5) and (F2) that a change in θ accompanied by a change in τ^k that keeps q^k constant, changes q^{jk} by

$$\begin{aligned} dq^{jk} &= \frac{d\tau^k}{1+r} + \frac{\gamma^{jk} d\theta}{1+g} \\ &= \frac{\left(\gamma^{jk} - \gamma^k\right) d\theta}{1+g}. \end{aligned}$$

As a result, the utility of a jk-mimicker will change according to

$$dv^{jk} = \frac{\partial v^{jk}}{\partial q^{jk}} dq^{jk} = -\alpha^{jk} d^k \frac{\left(\gamma^{jk} - \gamma^k\right) d\theta}{1+g}.$$

where α^{jk} denotes the jk-mimicker's marginal utility of income. Now if $\gamma^{jk} - \gamma^k > 0$ setting $d\theta > 0$ implies that $dv^{jk} < 0$ and if $\gamma^{jk} - \gamma^k < 0$ setting $d\theta < 0$ implies that $dv^{jk} < 0$. Either way, the jk-mimicker can be made worse off allowing a Pareto-improving move.

The upshot of this discussion is that if $\gamma^{jk} - \gamma^k > 0$ a reform that sets $d\theta > 0$ and changes q^{jk} according to the above relationship will make the jk-mimicker worse off and allows a Pareto-improving move. On the other hand, if $\gamma^{jk} - \gamma^k < 0$ a reform that sets $d\theta < 0$ allows a Pareto-improving move. Consequently, given this information structure, fiscal policy becomes overarching and one would want to either keep inflating the economy or deflating it. Now, given the pattern of binding self-selection constraint, the relevant sign for us is that of $\gamma^{h\ell} - \gamma^\ell$ which is negative (based on determinants of γ). Consequently, a deflationary reform of the type described always increases welfare, resulting in the optimality of the Friedman rule as a limit solution due to the constraint on the non-negativity of the nominal interest rate.

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